ARTICLE IN PRESS JAL:335

Journal of Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

The categorical imperative: Category theory as a foundation for deontic logic

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A R T I C L E I N F O

Article history:
Received 15 December 2013
Accepted 5 June 2014
Available online xxxx

Keywords:
Categorical logic
Deductive system
Conditional normative reasoning
Conflicting obligations
Paradoxes

A B S T R A C T

This article introduces a deontic logic which aims to model the Canadian legal discourse. Category theory is assumed as a foundational framework for logic. A deontic deductive system $\mathcal{DDS}$ is defined as two fibrations: the logic for unconditional obligations $\mathcal{OL}$ is defined within a Cartesian closed category on the grounds of an intuitionistic propositional action logic $\mathcal{PAL}$ and an action logic $\mathcal{AL}$, while a logic for conditional normative reasoning $\mathcal{CNR}$ is defined as a symmetric closed monoidal category. A typed syntax and typed arrows are used to define properly $\mathcal{DDS}$. We show how it can solve the paradoxes of deontic logic and we provide some examples of application to legal reasoning.

1. Introduction

Deontic logic was introduced in analogy with modal logic by von Wright [119] to model normative reasoning.[2] After the developments of possible world semantics with the work of Hintikka, Montague and Kripke (see [126]), von Wright's initial approach was redefined within the framework of modal logic. This gave rise to the well-known standard system of deontic logic, the modal logic *KD*. Many objections were raised against von Wright's initial approach,[3] but Chisholm's [38] paradox was the most damaging to the standard systems. It showed that monadic deontic logics cannot properly model conditional normative reasoning, which is central to the normative discourse.

Chisholm's objection was followed by various proposals. Some authors argued that contrary-to-duty reasoning should be modeled through a dyadic framework, specifying the conditions under which the obligations hold (see for example van Fraassen [118], Al-Hibri [2]).[4] Others argued that Chisholm's paradox arises because standard systems do not take into account the temporal dimension implicit to conditional

[1] The author would like to thank Jean-Pierre Marquis for valuable comments and discussions and anonymous referees for helpful comments on a previous draft of this paper. This research was financially supported by the Social Sciences and Humanities Research Council of Canada.

[2] See [46,83] for an introduction.

[3] See [83] for an overview.

[4] Historically, the building blocks of dyadic deontic logic were introduced by [120] in answer to Prior's [98] paradox of derived obligation. Dyadic deontic logic has however been used as a solution to Chisholm's puzzle.

http://dx.doi.org/10.1016/j.jal.2014.07.001
1570-8683/

normative reasoning (e.g., [43]), and this led to the introduction of temporal deontic logics (see for example Thomason [111], van Eck [116,117]). But still, other issues remained with these approaches, such as their inability to properly model conflicting obligations and factual detachment. This led some authors to introduce different solutions to answer the problems of detachment of deontic conditionals and, more importantly, to model conflicting conditional obligations. In addition to Makinson and van der Torre [80–82], who introduced input/output logics to model normative conditional and unconditional reasoning, one can find various proposals in non-monotonic (see [86] for an introduction) and adaptive logics (see for example [109,15,14]).

Even though deontic logic was first meant as a formal framework to analyze and evaluate normative reasoning, it has since then been used to serve different purposes. In addition to the analysis of inferences, deontic logic has been used to model normative systems (e.g., [76,35,21]) and multi-agent normative systems (e.g., [102,103,61,88,57,26]). It has been used to model contracts (e.g., [20,28,100]) and obligations with deadlines (e.g., [45,27,44]). It has been used in computer science (e.g., [125,114,34,33,19]), artificial intelligence (e.g., [24,23]) and in law (e.g., [104,124,97,54,18]).

This list does not pretend to be exhaustive and is in all likelihood incomplete. There is, however, a lesson that should be learned from this diversity: a system of deontic logic cannot be criticized independently of its purpose. For instance, a deontic logic which aims to model the evolution of a computer program will not require the same characteristics as one that tries to model the structure of the law. Similarly, a deontic logic that aims to model contracts will not need the same properties as one that models inferences. In the present paper, our aim is to introduce a deontic logic adequate to analyze and evaluate the structure of legal inferences. The aim is not to develop a deontic logic that can represent the structure of the Canadian legislation, nor to develop a formal system that can model how legal reasoning is done. Rather, our objective is to develop a formal system that can help in the analysis of the structure of legal reasoning, specifying how it should be done.

The results of the present paper are built upon previous work. Following the seminal work of Lambek [70] and what was presented in [95], we introduce a typed deontic logic within the framework of categorical logic. There are three main theoretical motivation for this paper. First, our aim is to introduce an alternative framework to modal logic to model unconditional obligations. Secondly, we wish to introduce an alternative framework to Boolean algebras to model actions within deontic contexts. Finally, the objective is to introduce a monadic formal system able to model conditional normative reasoning and conflicting obligations without requiring the techniques of non-monotonic or adaptive logics. As such, the deontic logic we propose will operate at three different levels. In a nutshell, we propose to define a deontic deductive system $\mathcal{DDS}$ on the grounds of an action logic $\mathcal{AL}$ and a propositional action logic $\mathcal{PAL}$ (cf. [90]), an obligation logic $\mathcal{OL}$ (cf. [94]) to model unconditional obligations and a logic $\mathcal{CNR}$ that can model conditional normative reasoning with a monadic O (cf. [93]).

The structure of the paper is as follows. In the next section, we present the rationale of our framework and expose the characteristics that a deontic logic which aims to model the Canadian legal discourse should satisfy. This will be followed by a brief exposition of the foundational framework we adopt. Then, in Section 3, we present all the relevant material that is required to define $\mathcal{DDS}$, and we provide the categorical definition in Section 4. The semantics is presented in Section 5, and a comparison of $\mathcal{DDS}$ with Goble's [51] analysis is provided in Section 6. Then, a discussion of some paradoxes is provided in Section 7, where we provide examples of applications of $\mathcal{DDS}$ to the Canadian legal discourse. We conclude in Section 8 with remarks for future research.

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2. Preliminaries

2.1. Some characteristics

The purpose of the present paper is to develop a formal framework relevant to the analysis of the Canadian legal discourse. As such, it ought to be founded on some of the Canadian legal discourse's characteristics. A first thing to mention is that *truth*, in Canadian law, is a matter of construction. Following the honorable Jean-Louis Baudoin [13, p. 5], the truth of normative propositions depends upon what is established by the legislator and the case law. In the deontic logic literature, this is consistent with Alchourrón's and Bulygin's [4, pp. 97, 102] conception, for whom obligations depend on norms, which are established by legal authorities. Hence, we assume that normative propositions are declarative statements, but which are not of the same type as descriptive propositions. This difference of type between normative and descriptive declarative sentences can be understood in the light of the semantical dichotomy between facts and norms (cf. Hume [62], Poincaré [96] and Jørgensen [66]). The semantical dichotomy implies that normative and descriptive propositions are not true in the same conditions. As such, an inference from (solely) normative propositions to a descriptive proposition will always be invalid, and similarly for inferences that go from (solely) descriptive propositions to a normative proposition.

From a legal point of view, *actions*, and not *propositions*, are obligatory, permitted or forbidden. Therefore, a deontic logic that wants to adequately model the legal discourse must be of the type *ought-to-do* rather than *ought-to-be* (see [122] for the distinction). Such a deontic logic must be founded upon a proper action logic able to model *human* actions. Its deontic operators will be of the type that takes an action and transforms it into a normative proposition. Iteration of deontic operators will thus be impossible.

Assuming that the truth of a normative proposition depends upon a norm established by some authority, it follows that tautologous actions are not obligatory (although, as we will see, there will be no such thing as 'tautologous' actions within $\mathcal{AL}$). As a result, the formal system must not satisfy formulas such as $\vdash O\top$ or $\vdash \varphi \supset O\top$, with $\top$ some tautologous action. In the literature, this position is consistent with Chellas [37], who argued that it must be possible to model a situation where noting is obligatory (not even tautologous actions), and with Jones and Pörn [65], who argued that it is always possible for someone to act against one's obligations.

A last characteristic worth mentioning is the presumption of consistency of the Canadian legal discourse. Of course, there are contradictory laws, but recall that our aim is not to model the structure of the legislation: our aim is rather to model the structure of the inferences we do with this (potentially contradictory) legislation. As it happens, even though there can be *a priori* contradictions within the law, it remains that the legal discourse is presupposed to be consistent *after* interpretation. The presumption of consistency is considered as a fundamental value of our judicial system, and as such the preservation of the legal discourse's consistency is a principle that guides the interpretation of the law [41, p. 387].

That being said, the presumption of consistency is not only admitted within the law literature but is also recognized by the Supreme Court of Canada as both a fundamental principle and a rational criterion for the legal discourse. On the one hand, it is a fundamental principle given that it justifies other principles that govern the interpretation of the law, such as the rule *nemo intelligere possit antequam iterum perlegerit*. This rule, which means that no one can understand a part without reading and rereading the whole in full, is meant to insure a consistent interpretation of the law.[5] On the other hand, consistency has been recognized by the Supreme Court as a criterion that enables us to evaluate the legal discourse's rationality. Quoting Sullivan [110, p. 176], the Supreme Court of Canada stated that the presumption of consistency is a "virtually irrebuttable" principle that insures that the legal discourse forms a rational framework.[6]

[5] See *2747-3174 Québec Inc. c. Québec (Régie des permis d'alcool)*, [1996] 3 RCS 919, paragraphe 207.
[6] *2747-3174 Québec Inc. c. Québec (Régie des permis d'alcool)*, [1996] 3 RCS 919, paragraphe 209.

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2.2. Foundational framework

In this paper, we follow the foundational framework adopted in [90]. We partially follow Lambek [71,72] in his categorical understanding of logic and assume category theory as a foundational framework to analyze the proof theory of different systems of logic.[7] In order to make this paper self-sufficient, we expose the relevant material to construct deductive systems and we refer the reader to the aforementioned papers for the technical details and the categorical definitions.[8]

From an epistemological point of view, category theory offers a powerful foundational framework that enables us to classify different systems of logic through their proof theory. A *deductive system* is understood as a (free) category $\mathcal{C}$ whose objects are formulas and whose arrows are proofs (deductions). The composition of arrows is represented by the rule (cut) and the deduction of a formula from itself is represented by the identity axiom (1).[9] An arrow represents a consequence relation or a deduction. We omit the names of the arrows for visual clarity.

$$\frac{\varphi \longrightarrow \psi \qquad \psi \longrightarrow \rho}{\varphi \longrightarrow \rho}\,(\text{cut}) \qquad\qquad \frac{}{\varphi \longrightarrow \varphi}\,(1)$$

The interest of the categorical understanding of logic appears when one considers how to introduce different logical connectives, with respect to the categorical structure of the deductive system. It is noteworthy that the connectives do not appear randomly when one defines a logical system from the proof-theoretical perspective of category theory. They appear in a specific order, and the properties of the logical system will actually depend upon the specific properties of some connectives.

From a deductive system, one can construct a *monoidal deductive system* by adding a tensor product $\otimes$ with a unit $\top$, who behave according to the rules (t) and (a) for the associativity of the tensor product, and (l) and (r) to make $\top$ into the unit of $\otimes$.[10]

$$\begin{array}{c} \varphi \longrightarrow \top \otimes \psi \\ \hline\hline \varphi \longrightarrow \psi \end{array}\,(\text{l}) \qquad\qquad \begin{array}{c} \varphi \longrightarrow \psi \otimes \top \\ \hline\hline \varphi \longrightarrow \psi \end{array}\,(\text{r})$$

$$\frac{\varphi \longrightarrow \psi \qquad \rho \longrightarrow \tau}{\varphi \otimes \rho \longrightarrow \psi \otimes \tau}\,(\text{t}) \qquad\qquad \begin{array}{c} \tau \longrightarrow (\varphi \otimes \psi) \otimes \rho \\ \hline\hline \tau \longrightarrow \varphi \otimes (\psi \otimes \rho) \end{array}\,(\text{a})$$

A *closed monoidal deductive system* is obtained on the grounds of a monoidal deductive system by introducing a right adjoint functor to the tensor product via the rule (cl). This rule is an analogue to the deduction theorem. Since in a closed monoidal deductive system the tensor product is not commutative, it follows that it can possess two different right adjoints. Having these connectives at our disposition, it is possible to introduce a special object $\bot$ to define intuitionistic negations via $\neg\varphi =_{def} \varphi \rhd \bot$ and $\sim \varphi =_{def} \varphi \blacktriangleright \bot$. In such a situation, we speak of a *closed monoidal deductive system with negation(s)*. If one wants the negations to behave classically, then one simply has to add the axioms $(\neg \sim)$ and $(\sim \neg)$ to

[7] See also [73].
[8] These definitions are listed in Appendix A.
[9] The rule (cut) respects associativity, i.e., $(hg)f = h(gf)$, while (1) respects the identity laws, i.e., given $f : \varphi \longrightarrow \psi$ and $g : \psi \longrightarrow \rho$, $f = 1_\psi f$ and $g = g1_\psi$.
[10] A double line means that the rule can be applied both ways.

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obtain a *closed deductive system with classical negations.*[11]

$$\frac{}{\neg \sim \varphi \longrightarrow \varphi}\ (\neg \sim) \qquad \frac{}{\sim \neg\varphi \longrightarrow \varphi}\ (\sim \neg)$$

$$\frac{\varphi \otimes \psi \longrightarrow \rho}{\varphi \longrightarrow \psi \triangleright \rho}\ (\text{cl}) \qquad \frac{\varphi \otimes \psi \longrightarrow \rho}{\psi \longrightarrow \varphi \blacktriangleright \rho}\ (\text{cl'})$$

A commutative tensor product is obtained by adding a braiding rule (b), which is its own inverse.[12] We thus obtain a *symmetric deductive system.*[13] If the deductive system is symmetric, then (r) can be proven from (l), and vice versa, as for (cl) and (cl'). Hence, in a symmetric closed deductive system, $\blacktriangleright$ and $\sim$ can be reduced to $\triangleright$ and $\neg$.

$$\frac{\varphi \longrightarrow \psi \otimes \tau}{\varphi \longrightarrow \tau \otimes \psi}\ (\text{b})$$

From a symmetric closed deductive system with classical negation, one can define a *compact deductive system* by adding the axioms (cpt1) and (cpt2).[14]

$$\frac{}{\varphi \triangleright \psi \longrightarrow \varphi^* \otimes \psi}\ (\text{cpt1}) \qquad \frac{}{\varphi^* \otimes \psi \longrightarrow \varphi \triangleright \psi}\ (\text{cpt2})$$

A *Cartesian deductive system* is a symmetric deductive system where the tensor product is a categorical product and its unit is a terminal object. This is formally represented by the rule (Cart) and the axiom (!).

$$\frac{\varphi \longrightarrow \psi \qquad \varphi \longrightarrow \rho}{\varphi \longrightarrow \psi \otimes \rho}\ (\text{Cart}) \qquad \frac{}{\varphi \longrightarrow \top}\ (!)$$

The co-tensor $\oslash$ is obtained through the opposite category $\mathcal{C}^{op}$ by reversing the arrows and replacing $\otimes$ and $\top$ respectively by $\oslash$ and $\bot$ to obtain the rules (co-t), (co-a), (co-l), (co-r) and (co-b). We thus speak of a *deductive system with co-tensor* (resp. co-product for Cartesian deductive systems). In the case of a Cartesian deductive system with co-product, the axiom and the rule are (0) and (co-Cart).

$$\frac{\varphi \longrightarrow \rho \qquad \psi \longrightarrow \rho}{\varphi \oslash \psi \longrightarrow \rho}\ (\text{co-Cart}) \qquad \frac{}{\bot \longrightarrow \varphi}\ (0)$$

The upshot of this mode of presentation is that the logical properties of a deductive system mainly depend upon the properties of the tensor product and the definition of negation. See [90,93] for a proper presentation and a comparison between deductive systems and the literature. For the purpose of this article, simply note that:

1. the multiplicative fragment of linear logic **MLL** introduced by [49] corresponds to a closed symmetric deductive system with classical negation and co-tensor;

[11] With $\varphi \cong \neg \sim \varphi \cong \sim \neg\varphi$ a natural isomorphism.
[12] Otherwise the deductive system would only be braided instead of symmetric.
[13] If the symmetric deductive system is closed, then there is only one adjoint to the tensor product (and incidentally one negation).
[14] In this case, the unit is the dualizing object (see [11] for the definition of a dualizing object).

2. intuitionistic logic is a closed Cartesian deductive system with negation and co-product;
3. classical logic is a closed Cartesian deductive system with classical negation and co-product.

3. Deontic deductive systems

The deontic logic we propose is called a *deontic deductive system*, $\mathcal{DDS}$ for short. It operates at three different levels, which were previously discussed at length in other papers. It incorporates an action logic, a logic for reasoning with unconditional obligations and a logic to reason with contrary-to-duties and conflicting obligations. In what follows, we only present what is relevant for the construction of $\mathcal{DDS}$ and we refer the reader to these articles for further details, explanations and discussions.

In what follows, we use a typed syntax and specify the types for logical connectives, propositions and consequence relations. This will be done to avoid some technical problems when trying to incorporate both a classical negation and an intuitionistic one within $\mathcal{DDS}$. This solution follows the ideas presented in [95] and is inspired by the work of Lambek [70]. Instead of using a unique arrow to model the consequence relation of our deductive system, we will use different types of arrows within different categories.

The introduction of typed arrows is necessary to incorporate both an intuitionistic negation and a classical one within $\mathcal{DDS}$. Otherwise, since initial objects are identical up to isomorphism, the classical negation would absorb the intuitionistic one. Although we could use an intuitionistic logic with strong negation (or equivalently a Nelson algebra, see [22,108]), this would require that we consider the strong negation as a primitive, which would imply that we step outside of our conceptual framework and define the strong negation without using the structural properties of our deductive system. As such, the typed syntax will allow us to stay within the framework of categorical logic.

3.1. Action logic

The first step consists in defining a proper action logic to represent the structure of human actions. It was argued in [90] that *choice* and *iteration*, although relevant for programing, are not primitive human actions. On the one hand, even though one may face a choice between two actions, one is not *performing* the action 'choice'. One will *choose*, and then ideally perform one of these actions, but one will not be 'choice-ing'. On the other hand, iteration in programing languages is meant to model recursive programs, and this is not relevant to human actions. Iteration of human actions can be represented by finite sequences, and as such we do not need iteration as a primitive. In addition to atomic actions, the two primitive complex human actions we assume are joint actions $\alpha \bullet \beta$ and sequence of actions $\alpha \curvearrowright \beta$.

Following [90], we distinguish between an *action logic* $\mathcal{AL}$, which represents the structure of actions, and a *propositional action logic* $\mathcal{PAL}$, expressing the structure of the language we use to talk about actions. Assume a category where arrows are proofs and objects are actions. Let Act be the collection of all atomic actions a_i. The well-formed formulas $WFF_{\mathcal{AL}}$ of type act are defined recursively from the language $\mathcal{L}_{\mathcal{AL}} = \{Act, \bullet, *, \ominus, \curvearrowright, (,)\}$ by[15]:

$$\alpha := a_i \mid * \mid \alpha \bullet \beta \mid \alpha \ominus \beta \mid \alpha \curvearrowright \beta$$

The connectives of $\mathcal{AL}$ are of the type that takes two actions and transforms them into another action (i.e., of type $\mathsf{act}\backslash\mathsf{act}/\mathsf{act}$).

Definition 1. An *Action Logic* $\mathcal{AL}$ is defined on the grounds of $\mathcal{L}_{\mathcal{AL}}$, $WFF_{\mathcal{AL}}$ and $\xrightarrow[\mathsf{AL}]{}$ arrows by the two following fragments.

[15] Note that in [90] $\otimes$ is used instead of $\bullet$. Here, we reserve the use of $\otimes$ for the analogue to the multiplicative conjunction in $\mathcal{CNR}$.

1. The fragment $\{Act, \curvearrowright, *, (,)\}$ is axiomatized by a monoidal deductive system (with $*$ the unit of $\curvearrowright$).
2. The fragment $\{Act, \bullet, \ominus, *, (,)\}$ is axiomatized by a compact deductive system, with:
   (a) $*$ the unit of $\bullet$ (which also dualizes);
   (b) $\alpha^* =_{def} * \ominus \alpha$.

$$\frac{}{\alpha^* \bullet \beta \xrightarrow[\mathsf{AL}]{} \beta \ominus \alpha}\ (\text{cpt1}) \qquad \frac{}{\beta \ominus \alpha \xrightarrow[\mathsf{AL}]{} \alpha^* \bullet \beta}\ (\text{cpt2}) \qquad \frac{\alpha \bullet \beta \xrightarrow[\mathsf{AL}]{} \gamma}{\beta \xrightarrow[\mathsf{AL}]{} \gamma \ominus \alpha}\ (\text{cl})$$

The connective $\bullet$ stands for action conjunction and is understood similarly to the multiplicative conjunction of linear logic (cf. [49]). It is the 'with' connective, $\alpha \bullet \beta$ meaning the action 'α together with β'. In an action logic, actions are *not* understood as declarative sentences. The conjunctive action $\alpha \bullet \beta$ stands for the simultaneous action of α and β, and as such it does not satisfy (Cart).

The connective $\ominus$ is introduced as the right adjoint of $\bullet$. From (cpt1) and (cpt2), it can be defined from $\bullet$ since we have an isomorphism between $\beta \ominus \alpha$ and $\alpha^* \bullet \beta$. It is not, however, interpreted similarly to the linear implication. The formula $\alpha \ominus \beta$ stands for the action 'α without β'. We wrote $\gamma \ominus \alpha$ instead $\alpha \ominus \gamma$ in (cl) to keep the intuitive reading of 'γ without α'. We can think of 'with' and 'without' informally as addition and subtraction: as a complex conjunctive action can be obtained by gluing actions together, an action can also be obtained by removing some parts of a more complex action.[16]

The connective $\curvearrowright$ stands for action sequence, the formula $\alpha \curvearrowright \beta$ meaning 'α and then β'. While action conjunction implies simultaneity, action sequence implies an order. Hence, the tensor product $\bullet$ is both associative and commutative, but sequence $\curvearrowright$ is only associative.

The special action $*$ is the *no change* or *nothing* action. As we will see in the next paragraphs, it is also an action that cannot be done (i.e., it is impossible to perform). Action negation is something which is often introduced without further ado within the literature. Unfortunately, though, we will not propose a thorough analysis of the concept within the present paper since it is beyond its scope (see [90] for details). Action negation is represented by the action α^* and it is understood as some form of complement of α.

An action is minimally understood as something which has causal powers, that is, something that can bring changes in the world. This understanding is uncontroversial within the literature (cf. Goldman [53], Pörn [102], Davidson [42], von Wright [121], Mossel [84]). As such, *negative* actions are actions too, given that not-acting will have repercussions in the world.[17]

That said, we understand a negative action as something which is either simply not done (omitted) or intentionally not done (forborne). Hence, the complement of an action can be seen as an absence of action (either intentional or not). Action negation is understood as the complement of an action in regards to the *no change* action. From the aforementioned definition, the dual action of α is '*no change* without α'.

An interesting consequence of defining an action logic as a compact deductive system is that we get $\alpha^* \bullet \beta$ isomorphic to $\beta \ominus \alpha$. This result is actually quite plausible: the action 'going to work without wearing a tie' is isomorphic to 'going to work while not wearing a tie'. Another consequence of this definition is that we have $*$ logically equivalent to $\alpha \bullet \alpha^*$. From this we can see why $*$ is an action that cannot be accomplished: it would be impossible to do both an action and its complement at the same time. As such, $*$ can informally be understood as something that does not have any causal powers. For instance, if we understand the evolution of a scenario as a discrete process, $*$ can be understood as the vacuous moment between the states.

At this point, the reader might be struck be the schizophrenic character of $*$: it appears that $*$ is both trivial (no change) and impossible (it cannot be performed). From a technical point of view, this is a

[16] Note that the 'without' action is part of our natural language and is often used to refer to actions that are somewhat *less* than other actions.

[17] Although all of these authors agree that an action has causal powers, they do not agree, however, on the meaning of action negation. See [90] for a discussion.

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consequence of defining an action logic as a compact deductive system. To show this, let us consider a Boolean algebra of actions (e.g., [107]). Usually, 0 is understood as the impossible action while 1 is the universal or trivial action. In such an algebra, 0 is initial while 1 is terminal. Although these properties are not satisfied in a non-Cartesian deductive system (e.g., if one defines an action logic as a symmetric closed deductive system with classical negation), one might nonetheless expect to keep the interpretation of 0 and 1 as the impossible and the trivial actions. That being said, it happens that in a compact deductive system $0 \cong 1$. As a result, writing $*$ instead of 1, we obtain that $*$ is at the same time something trivial but impossible.

Fortunately, we can provide a satisfying philosophical interpretation of this phenomenon and reconcile these interpretations. The special action $*$ is interpreted as something that does not have any causal powers. Even though we assumed that an action is something that has causal powers, and therefore in this respect $*$ is not an action per se, one can nonetheless see why $*$ is trivial: since it does not have any repercussions, it won't matter if it takes place together with or after an action. That said, assuming that $*$ does not have any causal powers, it also follows that $*$ is an action impossible to perform. Indeed, since either acting or not acting will have causal repercussions in the world, it follows that one cannot accomplish an action that will not have causal repercussions in the world. As such, $*$ is trivial but impossible to perform.

Having an action logic at our disposition, we can now define a propositional action logic $\mathcal{PAL}$, which uses declarative sentences. We argued in [90] that a propositional action logic should be defined as a closed Cartesian deductive system with negation and co-product, that is, as an intuitionistic deductive system. In short, the argument in favor of an intuitionistic formulation of $\mathcal{PAL}$ revolves around the fact that an action proposition can be considered as true only when we have a 'proof' that the action was actually done. This is consistent with the legal discourse. Not having a proof that the action was not done does not imply that it was done.

Let the collection of action propositions AP be defined by the following condition: if α is a well-formed formula of $\mathcal{AL}$ (of type act), then $\boldsymbol{\alpha}$ is an action proposition of type ap (for action propositions). The bold notation $\boldsymbol{\alpha}$ is used to refer to action propositions as declarative sentences.

The well-formed formulas $WFF_{\mathcal{PAL}}$ of type ap are defined recursively from $\mathcal{L}_{\mathcal{PAL}} = \{\mathsf{AP}, \wedge, \top, \supset, \vee, \bot, (,)\}$ by:

$$\varphi := \boldsymbol{\alpha} \mid \top \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \supset \psi$$

Definition 2. A *Propositional Action Logic* $\mathcal{PAL}$ is a closed Cartesian deductive system with negation and co-product built from $\mathcal{L}_{\mathcal{PAL}}$, $WFF_{\mathcal{PAL}}$ and $\underset{\mathsf{PAL}}{\longrightarrow}$ arrows. It has to satisfy the rule (Ψ).

$$\frac{\alpha \underset{\mathsf{AL}}{\longrightarrow} \beta}{\boldsymbol{\alpha} \underset{\mathsf{PAL}}{\longrightarrow} \boldsymbol{\beta}} \qquad (\Psi) \qquad \big(\text{with } \Psi(*) = \bot\big)$$

The rule (Ψ) enables us to import a part of the structure of $\mathcal{AL}$ into the atomic fragment of $\mathcal{PAL}$. It says that if two actions are related somehow, then there will also be a semantical relation within the language we use to talk about them.

3.2. Obligation logic

The second level at which $\mathcal{DDS}$ operates regards unconditional normative reasoning. By an unconditional normative reasoning, we mean an inference which deals with unconditional or *all things considered* obligations. These reasonings are modeled through an obligation logic $\mathcal{OL}$ inspired by previous work (cf. [94]).

Let $\mathcal{L}_{\mathcal{OL}} = \mathcal{L}_{\mathcal{AL}} \cup \mathcal{L}_{\mathcal{PAL}} \cup \{O, P_s\}$. The formulas of $\mathcal{OL}$ are of type np, for normative propositions. They are declarative sentences. The special propositions $\top$ and $\bot$ are either of type ap or np. Similarly, the

connectives $\wedge$, $\vee$ and $\supset$ are either of the type that takes two action propositions and transforms them into another action proposition (i.e., of type $\mathsf{ap}\backslash\mathsf{ap}/\mathsf{ap}$), or of the type that takes two normative propositions and transforms them into another normative proposition (i.e., of type $\mathsf{np}\backslash\mathsf{np}/\mathsf{np}$). They do not allow for mixed formulas.

Given α a formula of type act, the well-formed formulas $WFF_{\mathcal{OL}}$ of type np are defined recursively by:

$$\varphi := O\alpha \mid P_s\alpha \mid \top \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \supset \psi$$

The operators O and P_s stand respectively for obligation and strong permission. They are of a type that takes an action and transforms it into a normative proposition (i.e., of type np/act). The introduction of strong permission as a primitive operator is necessary to model permissions that are explicitly mentioned by a legal system. From O we define interdiction by (def. F) and weak permission by (def. P_w).

$$F\alpha =_{def} O\alpha^* \tag{def. F}$$

$$P_w\alpha =_{def} \neg F\alpha \tag{def. P_w}$$

The definition of weak permission in terms of 'not forbidden' is uncontroversial. If an action is not explicitly forbidden, then it is implicitly permitted. The definition of the F operator in terms of O is, however, sometimes contested.[18] Semantically, interdictions and obligations can be seen as two different things: an interdiction states that an action must *not* be performed, while an obligation states that an action *must* be performed. One says *act* and the other says *do not act*. As such, interdictions dictate the limits we must not trespass, while obligations stipulate the things we are required to do.

We disagree, however, with this semantical distinction. To see this, consider what is required to violate an interdiction and an obligation. In criminal law, there is a distinction between the *actus reus* (the act) and the *mens rea* (the intention). To determine if someone is imputable for a crime (or a fault), it must be established that the accused did indeed commit the crime (or the fault) and that his judgment was not clouded. It does not imply that the accused had the explicit intention to commit the crime (or the fault), but only that his mind was such that he was able to understand the consequences of his actions. Now, consider the two following propositions:

1. It is forbidden to not connect the lamp of a UV disinfection system.[19] ($F\alpha^*$)
2. A sign P-270 indicates that a car driver has the obligation to stop and yield to pedestrians on a pedestrian crossing.[20] ($O\beta$)

When does one violate the interdiction $F\alpha^*$? One violates this interdiction (in the context of sewage treatment) by 'not connecting the lamp of a UV disinfection system'. Hence, one violates $F\alpha^*$ by performing α^*. That said, α^* can be the result either of an omission or a forbearance, and one who does not connect the lamp of a UV disinfection system (in the context of sewage treatment) will be liable in both cases. On the other hand, one violates the obligation to yield to pedestrian at a pedestrian crossing when there is a sign P-270 when 'one does not yield to pedestrians' in that context. As such, one violates $O\beta$ by performing β^*. Hence, it is possible to argue that there is no fundamental semantical distinction between obligations and interdictions insofar as in the end, they mean the same thing: $O\alpha$ means that α^* should not be performed, as does $F\alpha^*$. Therefore, since both $O\alpha$ and $F\alpha^*$ mean the same thing and are violated in the same conditions, we assume that one can be defined in terms of the other.

[18] See for instance [58] and [40] for a discussion. See also [89,88,25].
[19] Règlement sur l'évacuation et le traitement des eaux usées des résidences isolées (*regulation on sewage for isolated residences*), RRQ, c Q-2, r 22.
[20] Règlement sur la signalisation routière (*regulation on road sign*), RRQ, c C-24.2, r 41, A.M. 99-06-15, a. 38.

Please cite this article in press as: C. Peterson, The categorical imperative: Category theory as a foundation for deontic logic, Journal of Applied Logic (2014), http://dx.doi.org/10.1016/j.jal.2014.07.001

Definition 3. $\mathcal{OL}$ is defined by a closed Cartesian deductive system with classical negation, co-product and arrows of type $\underset{\mathsf{OL}}{\longrightarrow}$. $\mathcal{OL}$ has to satisfy the axioms (**D**) and (**P**), and the rules (Δ) and (Π).

$$\frac{}{O\alpha \underset{\mathsf{OL}}{\longrightarrow} P_s\alpha} \quad (\mathbf{D}) \qquad\qquad \frac{}{P_s\alpha \underset{\mathsf{OL}}{\longrightarrow} \neg O\alpha^*} \quad (\mathbf{P})$$

$$\frac{\boldsymbol{\alpha} \underset{\mathsf{PAL}}{\longrightarrow} \boldsymbol{\beta}}{O\alpha \underset{\mathsf{OL}}{\longrightarrow} O\beta} \quad (\Delta) \quad \big(\text{with } \Delta(\bot) = \bot\big)$$

$$\frac{\boldsymbol{\alpha} \underset{\mathsf{PAL}}{\longrightarrow} \boldsymbol{\beta}}{P_s\alpha \underset{\mathsf{OL}}{\longrightarrow} P_s\beta} \quad (\Pi) \quad \big(\text{with } \Pi(\bot) = \bot\big)$$

The rules (Δ) and (Π) allow to represent deontic consequence and can be used to model deontic detachment.[21] They insure that if two actions are semantically related within $\mathcal{PAL}$, then the obligations (resp. strong permissions) will also be related in $\mathcal{OL}$. Note that $\Delta(\top)$ and $\Pi(\top)$ are undefined. From these rules and (Ψ), one can derive the rules (Σ_O) and (Σ_P) that allow the substitution of actions within the scope of O and P_s. Considered together, these rules preserve the structure of $\mathcal{AL}$ throughout $\mathcal{PAL}$ and $\mathcal{OL}$. They insure that the three deductive systems behave coherently.

$$\frac{\alpha \underset{\mathsf{AL}}{\longrightarrow} \beta}{O\alpha \underset{\mathsf{OL}}{\longrightarrow} O\beta} \quad (\Sigma_O) \quad \big(\text{with } \Sigma_O(*) = \bot\big)$$

$$\frac{\alpha \underset{\mathsf{AL}}{\longrightarrow} \beta}{P_s\alpha \underset{\mathsf{OL}}{\longrightarrow} P_s\beta} \quad (\Sigma_P) \quad \big(\text{with } \Sigma_P(*) = \bot\big)$$

Axiom (**D**) states that if an action is obligatory, then it is strongly (explicitly) permitted, and axiom (**P**) expresses that if an action is strongly permitted, then it is weakly (implicitly) permitted. Together with (cut), axioms (**D**) and (**P**) allow us to recover the axiom schema (D) of standard deontic logic. From (D), the definition of negation and (cl), we recover the theorem for normative consistency (NC). This theorem is necessary to adequately model the legal discourse, since, as we saw in Section 2.1, the presumption of consistency of the legal discourse (after interpretation) is considered by the Supreme Court of Canada as a criterion of its rationality. As a result, a deontic logic which aims to model the Canadian legal discourse must validate the presumption of consistency for unconditional obligations. An action and its complement cannot be obligatory at the same time and under the same conditions.

$$O\alpha \underset{\mathsf{OL}}{\longrightarrow} \neg O\alpha^* \tag{D}$$

$$O\alpha \wedge O\alpha^* \underset{\mathsf{OL}}{\longrightarrow} \bot \tag{NC}$$

3.3. Conditional normative reasoning

The last step is to introduce the appropriate material to model conditional normative reasoning and conflicting obligations. In [93], the logic $\mathcal{CNR}$ was presented as a foundational framework that can model conditional normative reasoning through a monadic O without requiring further operators. The properties and the virtues of $\mathcal{CNR}$ as a foundational framework were analyzed thoroughly and we refer the reader to the paper for further details, explanations and discussion.

[21] That is, if $O\alpha$ and $\boldsymbol{\alpha} \longrightarrow \boldsymbol{\beta}$, then $O\beta$.

Let *Prop* be a collection of atomic descriptive (declarative) formulas p_i of type d. The language for conditional normative reasoning is defined by:

$$\mathcal{L}_{\mathcal{CNR}} = \{(,), \mathsf{AP}, \mathit{Prop}, \otimes, 1, \multimap, \parr, 0, O, P_s\}$$

Assuming that α is a well-formed formula of type act, the well-formed formulas of $\mathit{WFF}_{\mathcal{L}_{\mathcal{CNR}}}$ are defined recursively by:

$$\varphi := 1 \mid 0 \mid \boldsymbol{\alpha} \mid p_i \mid O\alpha \mid P_s\alpha \mid \varphi \otimes \psi \mid \varphi \multimap \psi \mid \varphi \parr \psi$$

As in $\mathcal{OL}$, the operators O and P_s are of type np/act. The main distinction between $\mathcal{L}_{\mathcal{CNR}}$ and $\mathcal{L}_{\mathcal{OL}}$ is that the former allows for mixed formulas. Although the propositions of $\mathcal{CNR}$ are of type np, the connectives $\otimes$, $\multimap$ and $\parr$ are either of the type $\mathsf{np}\backslash\mathsf{np}/\mathsf{np}$, $\mathsf{ap}\backslash\mathsf{np}/\mathsf{np}$, $\mathsf{np}\backslash\mathsf{np}/\mathsf{ap}$, $\mathsf{d}\backslash\mathsf{np}/\mathsf{np}$ or $\mathsf{np}\backslash\mathsf{np}/\mathsf{d}$. The special propositions 0 and 1 are of type np or ap. The connective $\otimes$ is similar to the multiplicative conjunction, $\multimap$ is similar to the linear implication and $\parr$ is similar to the multiplicative disjunction of linear logic.[22]

Given that $\mathcal{CNR}$'s structure is weaker than $\mathcal{OL}$'s, we need to modify $\mathcal{PAL}$ and define it as a symmetric closed category. Note that we still keep intuitionistic negation. Let $\mathcal{L}_{\mathcal{PAL}^{\otimes}} = \{(,), \mathsf{AP}, \otimes, 1, \multimap, \parr, 0\}$. The connectives of $\mathcal{PAL}^{\otimes}$ are of type $\mathsf{ap}\backslash\mathsf{ap}/\mathsf{ap}$. The well-formed formulas $\mathit{WFF}_{\mathcal{PAL}^{\otimes}}$ are defined recursively by:

$$\varphi := \boldsymbol{\alpha} \mid 1 \mid 0 \mid \varphi \otimes \psi \mid \varphi \parr \psi \mid \varphi \multimap \psi$$

Definition 4. $\mathcal{PAL}^{\otimes}$ is defined from $\mathcal{L}_{\mathcal{PAL}^{\otimes}}$ and $\mathit{WFF}_{\mathcal{PAL}^{\otimes}}$ by a closed symmetric deductive system with negation, co-tensor and $\underset{\mathsf{PAL}^{\otimes}}{\longrightarrow}$ arrows. It has to satisfy the rule $(\Psi_{\otimes})$.

$$\frac{\alpha \underset{\mathsf{AL}}{\longrightarrow} \beta}{\boldsymbol{\alpha} \underset{\mathsf{PAL}^{\otimes}}{\longrightarrow} \boldsymbol{\beta}} \quad (\Psi_{\otimes}) \qquad \big(\text{with } \Psi_{\otimes}(*) = \bot\big)$$

Definition 5. $\mathcal{CNR}$ is defined from $\mathcal{L}_{\mathcal{CNR}}$ and $\mathit{WFF}_{\mathcal{L}_{\mathcal{CNR}}}$ by a closed symmetric deductive system with classical negation, co-tensor and $\underset{\mathsf{CNR}}{\longrightarrow}$ arrows. Negation is defined by $\sim \varphi =_{def} \varphi \multimap 0$. It also has to satisfy the axioms $(\mathbf{D}_{\otimes})$ and $(\mathbf{P}_{\otimes})$ together with the rules $(\Delta_{\otimes})$ and $(\Pi_{\otimes})$.

$$\frac{}{O\alpha \underset{\mathsf{CNR}}{\longrightarrow} P_s\alpha} \; (\mathbf{D}_{\otimes}) \qquad\qquad \frac{}{P_s\alpha \underset{\mathsf{CNR}}{\longrightarrow} \sim O\alpha^*} \; (\mathbf{P}_{\otimes})$$

$$\frac{\boldsymbol{\alpha} \underset{\mathsf{PAL}^{\otimes}}{\longrightarrow} \boldsymbol{\beta}}{O\alpha \underset{\mathsf{CNR}}{\longrightarrow} O\beta} \quad (\Delta_{\otimes}) \qquad \big(\text{with } \Delta_{\otimes}(\bot) = \bot\big)$$

$$\frac{\boldsymbol{\alpha} \underset{\mathsf{PAL}^{\otimes}}{\longrightarrow} \boldsymbol{\beta}}{P_s\alpha \underset{\mathsf{CNR}}{\longrightarrow} P_s\beta} \quad (\Pi_{\otimes}) \qquad \big(\text{with } \Pi_{\otimes}(\bot) = \bot\big)$$

$\mathcal{CNR}$ and $\mathcal{PAL}^{\otimes}$ are the monoidal counterparts of $\mathcal{OL}$ and $\mathcal{PAL}$, which are defined as Cartesian deductive systems.[23] The rules and the axioms have the same purpose and interpretation. The definition of $\mathcal{CNR}$ as a symmetric closed deductive system enables us to model properly conditional normative reasoning and conflicting obligations.

[22] It would be wrong, however, to assume that we are working within the framework of linear logic. Thus defined, the propositional part of $\mathcal{CNR}$ (without the deontic operators) and the multiplicative fragment **MLL** of linear logic only share the same categorical structure. See [93] for details.

[23] Note that, strictly speaking, a Cartesian deductive system is also monoidal.

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Table 1
Summary of rules and axioms.

	Arrow	Language	Axioms	Rules
$\mathcal{AL}$	$\underset{\mathsf{AL}}{\longrightarrow}$		(1)	(cut)
		$\bullet$		(t), (a), (l), (b)
		$\ominus$		(cl), (cpt1), (cpt2)
		$*$	$(**)$	
		$\curvearrowright$		(t), (a), (l), (r)
$\mathcal{PAL}$	$\underset{\mathsf{PAL}}{\longrightarrow}$		(1)	(cut), (Ψ)
		$\wedge$		(Cart)
		$\supset$		(cl)
		$\vee$		(co-Cart)
		$\perp$	(0)	
		$\top$	(!)	
$\mathcal{PAL}^{\otimes}$	$\underset{\mathsf{PAL}^{\otimes}}{\longrightarrow}$		(1)	(cut), $(\Psi_{\otimes})$
		$\otimes$		(t), (a), (l), (b)
		$\multimap$		(cl)
		$\parr$		(co-t), (co-a), (co-l), (co-b)
$\mathcal{OL}$	$\underset{\mathsf{OL}}{\longrightarrow}$		(1)	(cut)
		$\wedge$		(Cart)
		$\supset$		(cl)
		$\vee$		(co-Cart)
		$\perp$	(0), $(\neg\neg)$	
		$\top$	(!)	
		O	$(\mathbf{D})$	(Δ)
		P_s	$(\mathbf{P})$	(Π)
$\mathcal{CNR}$	$\underset{\mathsf{CNR}}{\longrightarrow}$		(1)	(cut)
		$\otimes$		(t), (a), (l), (b)
		$\multimap$		(cl)
		$\parr$		(co-t), (co-a), (co-l), (co-b)
		0	$(\sim\sim)$	
		O	$(\mathbf{D}^{\otimes})$	$(\Delta_{\otimes})$
		P_s	$(\mathbf{P}^{\otimes})$	$(\Pi_{\otimes})$

Thus constructed, $\mathcal{CNR}$ is defined as an instance of a $*$-autonomous category (cf. [10,11]) satisfying extra conditions. As such, its rules and axioms are similar to those of the multiplicative fragment **MLL** of linear logic (cf. [49]). It has actually been shown by Seely [106] that the categorical structure of **MLL**, when interpreted as a free category, is precisely that of a $*$-autonomous category. He also showed that the categorical structure of the multiplicative and additive fragment **MALL**, that is, linear logic without exponentials, is a $*$-autonomous category with product and co-product. Despite the comparison that can be made between $\mathcal{CNR}$ and **MLL**, which are both instances of a closed symmetric deductive system with classical negation, we chose the acronym for *conditional normative reasoning* instead of *deontic multiplicative linear logic* to avoid any confusion with the deontic linear logic presented by Lokhorst [78].[24]

From the aforementioned deductive systems, we define a *deontic deductive system* $\mathcal{DDS}$ from $\mathcal{AL}$, $\mathcal{PAL}$, $\mathcal{PAL}^{\otimes}$, $\mathcal{OL}$ and $\mathcal{CNR}$. Table 1 above summarizes the rules and axioms of the deductive systems presented so far and Table 2 summarizes the types of each logical connective and specifies the arrow types that use them.

4. Categorical definition of $\mathcal{DDS}$

A deontic deductive system is defined on the grounds of four logics, which each have a different structure. The rules (Ψ), (Δ) and (Π) (resp. $(\Psi_{\otimes})$, $(\Delta_{\otimes})$ and $(\Pi_{\otimes})$) are meant to link these logics together and preserve the structure between actions throughout the other deductive systems. From a categorical perspective, these

[24] A comparison between $\mathcal{CNR}$ and deontic linear logic can be found in [93].

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Table 2
Summary of types.

	$\bullet, \ominus, \curvearrowright$	$*$	$\otimes, \multimap, \parr$	$1, 0$	$\wedge, \supset, \vee$	$\top, \bot$	O, P_s
$\mathcal{AL}$	act\act/act	act					
$\mathcal{PAL}$					ap\ap/ap	ap	
$\mathcal{OL}$					np\np/np	np	np/act
$\mathcal{PAL}^{\otimes}$			ap\ap/ap	ap			
$\mathcal{CNR}$			np\np/np ap\np/np d\np/np np\np/ap np\np/d	np			np/act

rules can be defined as functors that have specific properties. Indeed, they can be defined as fibrations between the atomic fragments of the deductive systems.[25]

The concept of a fibration is of both philosophical and technical importance. Historically, the notion of fibration (or fibred category) appeared in the work of Grothendieck [55] (cf. [32]) and was applied to algebraic geometry and homotopy theory. In addition to their applications in mathematics, fibrations (but more generally category theory) have been applied to computer science (see for example [12]) and have had considerable repercussions in logic and type theory, as we can see for instance in Jacob's [63] book. More recently, though, there have also been many developments in homotopy type theory, where the epistemological and technical significance of fibrations emerges from the relations between homotopy theory, type theory and computer science (see [101]).

Although the notion of fibration is not essential to our proposal, it remains that technically it provides an accurate formalization of the relations between $\mathcal{DDS}$'s fragments. Incidentally, it has some interesting epistemological ramifications. Assuming that $\mathcal{DDS}$ is an adequate formalization of our natural language, it suggests that the structure of our natural language might be more similar than we think with other mathematical structures.

From a logical perspective, fibrations allow to model indexing and substitution [63, p. 20], which is precisely the role played by (Ψ), (Δ) and (Π) (resp. $(\Psi_\otimes)$, $(\Delta_\otimes)$ and $(\Pi_\otimes)$) within $\mathcal{DDS}$'s definition. Indeed, these rules allow to import the structure of actions represented in $\mathcal{AL}$ into the atomic fragments of the other logics. Consider the following fragments.

Definition 6. $\mathcal{PAL}_\bigstar$ is the fragment of $\mathcal{PAL}$ where well-formed formulas are defined by:

$$\varphi := \boldsymbol{\alpha} \mid \bot$$

Definition 7. $\mathcal{OL}_\bigstar$ is the fragment of $\mathcal{OL}$ where well-formed formulas are defined by:

$$\varphi := O\alpha \mid \bot$$

The definition of (Ψ) as a fibration requires a little technical modification to $\mathcal{AL}$. Let us augment $\mathcal{AL}$ to $\mathcal{AL}^\delta$ with a dummy equivalence class of actions $[\delta]$, for which the action δ is the representative. The dummy action has to satisfy the axiom (0), and thus it is understood as a dummy initial object.

$$\frac{}{\delta \longrightarrow \alpha}\,(0)$$

The technical usefulness of δ appears when we consider how to define f Cartesian over 0_α (see the two diagrams below). The first option would be to assign an arbitrary $* \xrightarrow{f} \alpha$ Cartesian over 0_α and define

[25] See Appendix A for the definitions of a functor, a fibration and the property of being Cartesian over.

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$* \xrightarrow{g} \beta$ as an isomorphism. This option, however, is problematic. Firstly, this would turn $*$ into an initial object in $\mathcal{AL}$, and this would have devastating consequences. Indeed, anything would then be deducible from anything. Recall that $*$ is the unit of $\bullet$. We would have $* \longrightarrow \beta \ominus \alpha$ for any β and any α, hence by (cl) we would have $* \bullet \alpha \longrightarrow \beta$ and by (l) $\alpha \longrightarrow \beta$. The second option would be to define both $* \xrightarrow{f} \alpha$ and $* \xrightarrow{g} \beta$ as isomorphisms. However, this would imply that every action in $\mathcal{AL}$ is isomorphic to $*$, hence impossible to perform.

$$
\begin{tikzcd}
* \arrow[d, "1_*"'] \arrow[dr, "f"] & \\
* \arrow[r, "f"'] & \alpha
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta \arrow[d, "g^{-1}"'] \arrow[dr, "f \circ g^{-1}"] & \\
* \arrow[r, "f"'] & \alpha
\end{tikzcd}
$$

$$
\begin{tikzcd}
\perp \arrow[r, "0_\alpha"] & \boldsymbol{\alpha} \\
\perp \arrow[u, "1_\perp"] \arrow[ur, "0_\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \arrow[r, "0_\alpha"] & \boldsymbol{\alpha} \\
\boldsymbol{\beta} \arrow[u, "0_\beta^{-1}"] \arrow[ur, "0_\alpha \circ 0_\beta^{-1}"'] &
\end{tikzcd}
$$

The introduction of δ is meant to shadow the Cartesian structure induced by $\perp$ in $\mathcal{PAL}$. Roughly speaking, the idea is to sit the compact closed structure of $\mathcal{AL}$ on some dummy initial object δ. There is an obvious inclusion functor:

$$\mathcal{AL} \hookrightarrow \mathcal{AL}^\delta$$

Definition 8. $\mathcal{AL}^\delta \xrightarrow{\Psi} \mathcal{PAL}_\bigstar$ is a fibration. The functor is defined by:

$$\Psi(*) = \perp$$
$$\Psi(\delta) = \perp$$
$$\Psi(\alpha) = \boldsymbol{\alpha}$$

1. $\perp \underset{\mathsf{PAL}}{\longrightarrow} \boldsymbol{\alpha}$

$$
\begin{tikzcd}
\delta \arrow[d, "1_\delta"'] \arrow[dr, "0_\alpha"] & \\
\delta \arrow[r, "0_\alpha"'] & \alpha
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta \arrow[d, "0_\beta^{-1}"'] \arrow[dr, "0_\alpha \circ 0_\beta^{-1}"] & \\
\delta \arrow[r, "0_\alpha"'] & \alpha
\end{tikzcd}
$$

$$
\begin{tikzcd}
\perp \arrow[r, "0_\alpha"] & \boldsymbol{\alpha} \\
\perp \arrow[u, "1_\perp"] \arrow[ur, "0_\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \arrow[r, "0_\alpha"] & \boldsymbol{\alpha} \\
\boldsymbol{\beta} \arrow[u, "0_\beta^{-1}"] \arrow[ur, "0_\alpha \circ 0_\beta^{-1}"'] &
\end{tikzcd}
$$

2. $\boldsymbol{\alpha} \underset{\mathsf{PAL}}{\longrightarrow} \perp$

$$
\begin{tikzcd}
* \arrow[d, "f"'] \arrow[dr, "1_*"] & \\
\alpha \arrow[r, "f^{-1}"'] & *
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta \arrow[d, "f \circ g^{-1}"'] \arrow[dr, "g^{-1}"] & \\
\alpha \arrow[r, "f^{-1}"'] & *
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boldsymbol{\alpha} \arrow[r, "0_\alpha^{-1}"] & \perp \\
\perp \arrow[u, "0_\alpha"] \arrow[ur, "1_\perp"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{\alpha} \arrow[r, "0_\alpha^{-1}"] & \perp \\
\boldsymbol{\beta} \arrow[u, "0_\alpha \circ 0_\beta^{-1}"] \arrow[ur, "0_\beta^{-1}"'] &
\end{tikzcd}
$$

The first case deals with the fact that $\perp$ is initial in $\mathcal{PAL}$. It says that when $\boldsymbol{\beta}$ is isomorphic to $\perp$, β is isomorphic to δ.

The second case implies that $\boldsymbol{\alpha}$ is isomorphic to $\perp$. There is a Cartesian f^{-1} which makes α isomorphic to $*$, and if there is $\boldsymbol{\beta}$ isomorphic to $\perp$, then β will also be isomorphic to $*$. We define f^{-1} such that

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$f^{-1} \circ f = 1_*$ and $f \circ f^{-1} = 1_\alpha$, similarly for g^{-1}.

3. $\boldsymbol{\beta} \underset{\mathsf{PAL}}{\longrightarrow} \boldsymbol{\alpha}$

$$\begin{tikzcd} \delta \arrow[d, "0_\beta"'] \arrow[dr, "0_\alpha"] & \\ \beta \arrow[r, "f"'] & \alpha \end{tikzcd} \qquad \begin{tikzcd} \gamma \arrow[d, "h"'] \arrow[dr, "g"] & \\ \beta \arrow[r, "f"'] & \alpha \end{tikzcd}$$

$$\begin{tikzcd} \boldsymbol{\beta} \arrow[r, "\Psi(f)"] & \boldsymbol{\alpha} \\ \perp \arrow[u, "0_\beta"] \arrow[ur, "0_\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\beta} \arrow[r, "\Psi(f)"] & \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \arrow[u, "\Psi(h)"] \arrow[ur, "\Psi(g)"'] & \end{tikzcd}$$

4. $\perp \underset{\mathsf{PAL}}{\longrightarrow} \perp$

$$\begin{tikzcd} * \arrow[d, "1_*"'] \arrow[dr, "1_*"] & \\ * \arrow[r, "1_*"'] & * \end{tikzcd} \qquad \begin{tikzcd} \alpha \arrow[d, "f^{-1}"'] \arrow[dr, "f^{-1}"] & \\ * \arrow[r, "1_*"'] & * \end{tikzcd}$$

$$\begin{tikzcd} \perp \arrow[r, "1_\perp"] & \perp \\ \perp \arrow[u, "1_\perp"] \arrow[ur, "1_\perp"'] & \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r, "1_\perp"] & \perp \\ \boldsymbol{\alpha} \arrow[u, "0_\alpha^{-1}"] \arrow[ur, "0_\alpha^{-1}"'] & \end{tikzcd}$$

In the third case, we assume that if there is an arbitrary $\Psi(f)$, then it is the target of an arbitrary f. We use the dummy δ to define f Cartesian over $\Psi(f)$ for the case where $\perp$ is initial, otherwise we simply use the fact that Ψ is a functor.

In the fourth case, we assume that α is isomorphic to $*$ when $\boldsymbol{\alpha}$ is isomorphic to $\perp$.

Definition 9. $\mathcal{PAL}_\bigstar \xrightarrow{\Delta} \mathcal{OL}_\bigstar$ is a fibration defined by:

$$\Delta(\perp) = \perp$$
$$\Delta(\boldsymbol{\alpha}) = O\alpha$$

1. $\perp \underset{\mathsf{OL}}{\longrightarrow} O\alpha$

$$\begin{tikzcd} \perp \arrow[d, "1_\perp"'] \arrow[dr, "0_\alpha"] & \\ \perp \arrow[r, "0_\alpha"'] & \boldsymbol{\alpha} \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\beta} \arrow[d, "0_\beta^{-1}"'] \arrow[dr, "g"] & \\ \perp \arrow[r, "0_\alpha"'] & \boldsymbol{\alpha} \end{tikzcd}$$

$$\begin{tikzcd} \perp \arrow[r, "0_{O\alpha}"] & O\alpha \\ \perp \arrow[u, "1_\perp"] \arrow[ur, "0_{O\alpha}"'] & \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r, "0_{O\alpha}"] & O\alpha \\ O\beta \arrow[u, "0_{O\beta}^{-1}"] \arrow[ur, "\Delta(g)"'] & \end{tikzcd}$$

2. $O\beta \underset{\mathsf{OL}}{\longrightarrow} O\alpha$

$$\begin{tikzcd} \perp \arrow[d, "0_\beta"'] \arrow[dr, "0_\alpha"] & \\ \boldsymbol{\beta} \arrow[r, "f"'] & \boldsymbol{\alpha} \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\gamma} \arrow[d, "h"'] \arrow[dr, "g"] & \\ \boldsymbol{\beta} \arrow[r, "f"'] & \boldsymbol{\alpha} \end{tikzcd}$$

$$\begin{tikzcd} O\beta \arrow[r, "\Delta(f)"] & O\alpha \\ \perp \arrow[u, "0_{O\beta}"] \arrow[ur, "0_{O\alpha}"'] & \end{tikzcd} \qquad \begin{tikzcd} O\beta \arrow[r, "\Delta(f)"] & O\alpha \\ O\gamma \arrow[u, "\Delta(h)"] \arrow[ur, "\Delta(g)"'] & \end{tikzcd}$$

3. $O\alpha \underset{\mathsf{OL}}{\longrightarrow} \perp$

$$\begin{tikzcd} \perp \arrow[d, "0_\alpha"'] \arrow[dr, "1_\perp"] & \\ \boldsymbol{\alpha} \arrow[r, "0_\alpha^{-1}"'] & \perp \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\beta} \arrow[d, "f"'] \arrow[dr, "0_\beta^{-1}"] & \\ \boldsymbol{\alpha} \arrow[r, "0_\alpha^{-1}"'] & \perp \end{tikzcd}$$

$$\begin{tikzcd} O\alpha \arrow[r, "0_{O\alpha}^{-1}"] & \perp \\ \perp \arrow[u, "0_{O\alpha}"] \arrow[ur, "1_\perp"'] & \end{tikzcd} \qquad \begin{tikzcd} O\alpha \arrow[r, "0_{O\alpha}^{-1}"] & \perp \\ O\beta \arrow[u, "\Delta(f)"] \arrow[ur, "0_{O\beta}^{-1}"'] & \end{tikzcd}$$

4. $\perp \underset{\mathsf{OL}}{\longrightarrow} \perp$

$$\begin{tikzcd} \perp \arrow[d, "1_\perp"'] \arrow[dr, "1_\perp"] & \\ \perp \arrow[r, "1_\perp"'] & \perp \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\beta} \arrow[d, "0_\beta^{-1}"'] \arrow[dr, "0_\beta^{-1}"] & \\ \perp \arrow[r, "1_\perp"'] & \perp \end{tikzcd}$$

$$\begin{tikzcd} \perp \arrow[r, "1_\perp"] & \perp \\ \perp \arrow[u, "1_\perp"] \arrow[ur, "1_\perp"'] & \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r, "1_\perp"] & \perp \\ O\beta \arrow[u, "0_{O\beta}^{-1}"] \arrow[ur, "0_{O\beta}^{-1}"'] & \end{tikzcd}$$

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Definition 10. $\mathcal{PAL}_\bigstar \xrightarrow{\Pi} \mathcal{OL}_\bigstar$ is a fibration defined by:

$$\Pi(\perp) = \perp$$

$$\Pi(\boldsymbol{\alpha}) = P_s\alpha$$

The definition goes along the lines of Definition 9 (replace O by P_s).

Since the composition of two fibrations yields another fibration (see Lemma 1.5.5 of [63]), it follows that we also have substitution rules such that $\Sigma_O = \Delta \circ \Psi$ and $\Sigma_P = \Pi \circ \Psi$ that go directly from $\mathcal{AL}$ to $\mathcal{OL}_\bigstar$.

The definitions of the fibrations in the cases of the symmetric deductive systems are slightly different. Consider the following fragments.

Definition 11. $\mathcal{PAL}^\otimes_\bigstar$ is the atomic fragment of $\mathcal{PAL}^\otimes$ defined from the well-formed formulas:

$$\varphi := \boldsymbol{\alpha} \mid 0$$

Definition 12. $\mathcal{CNR}_\bigstar$ is the fragment of $\mathcal{CNR}$ where well-formed formulas are defined by:

$$\varphi := O\alpha \mid 0$$

Considering that 0 is not initial in $\mathcal{PAL}^\otimes$, we do not need the dummy δ to define $\Psi_\otimes$. Actually, the cost of keeping δ would be to induce $\mathcal{AL}$'s dummy structure into $\mathcal{PAL}^\otimes$, turning 0 into an initial object (which is undesirable, otherwise we would obtain deontic explosion).

Definition 13. $\mathcal{AL} \xrightarrow{\Psi_\otimes} \mathcal{PAL}^\otimes_\bigstar$ is a fibration defined by:

$$\Psi_\otimes(*) = 0$$

$$\Psi_\otimes(\alpha) = \boldsymbol{\alpha}$$

1. $0 \underset{\mathsf{PAL}^\otimes}{\longrightarrow} \boldsymbol{\alpha}$

$$\begin{tikzcd} * \arrow[d, "1_*"'] \arrow[dr, "f"] & \\ * \arrow[r, "f"'] & \alpha \\ 0 \arrow[r, "\Psi_\otimes(f)"] & \boldsymbol{\alpha} \\ 0 \arrow[u, "1_0"] \arrow[ur, "\Psi_\otimes(f)"'] & \end{tikzcd} \qquad \begin{tikzcd} \beta \arrow[d, "g^{-1}"'] \arrow[dr, "f\circ g^{-1}"] & \\ * \arrow[r, "f"'] & \alpha \\ 0 \arrow[r, "\Psi_\otimes(f)"] & \boldsymbol{\alpha} \\ \boldsymbol{\beta} \arrow[u, "\Psi_\otimes(g^{-1})"] \arrow[ur, "\Psi_\otimes(f\circ g^{-1})"'] & \end{tikzcd}$$

2. $\boldsymbol{\alpha} \underset{\mathsf{PAL}^\otimes}{\longrightarrow} 0$

$$\begin{tikzcd} * \arrow[d, "f"'] \arrow[dr, "1_*"] & \\ \alpha \arrow[r, "f^{-1}"'] & * \\ \boldsymbol{\alpha} \arrow[r, "\Psi_\otimes(f^{-1})"] & 0 \\ 0 \arrow[u, "\Psi_\otimes(f)"] \arrow[ur, "1_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \beta \arrow[d, "f\circ g^{-1}"'] \arrow[dr, "g^{-1}"] & \\ \alpha \arrow[r, "f^{-1}"'] & * \\ \boldsymbol{\alpha} \arrow[r, "\Psi_\otimes(f^{-1})"] & 0 \\ \boldsymbol{\beta} \arrow[u, "\Psi_\otimes(f\circ g^{-1})"] \arrow[ur, "\Psi_\otimes(g^{-1})"'] & \end{tikzcd}$$

In the first case, we define f^{-1} such that $f^{-1} \circ f = 1_*$ and $f \circ f^{-1} = 1_\alpha$, similarly for g^{-1}, when there is an arrow from 0 to $\boldsymbol{\alpha}$ (resp. to $\boldsymbol{\beta}$). We require that $\Psi_\otimes(f^{-1}) = \Psi_\otimes(f)^{-1}$, and similarly for $\Psi_\otimes(g^{-1})$. As such, if there is an arrow from 0 to $\boldsymbol{\alpha}$, then it is an isomorphism, making α isomorphic to $*$.

The same strategy is applied to the second case.

3. $\boldsymbol{\beta} \underset{\mathsf{PAL}^{\otimes}}{\longrightarrow} \boldsymbol{\alpha}$

$$\begin{tikzcd} * \arrow[d, "g"'] \arrow[dr, "f"] & \\ \beta \arrow[r, "f\circ g^{-1}"'] & \boldsymbol{\alpha} \\ \boldsymbol{\beta} \arrow[r, "\Psi_\otimes(h)"] & \boldsymbol{\alpha} \\ 0 \arrow[u, "\Psi_\otimes(g)"] \arrow[ur, "\Psi_\otimes(f)"'] & \end{tikzcd} \qquad \begin{tikzcd} \gamma \arrow[d, "g"'] \arrow[dr, "f"] & \\ \beta \arrow[r, "h"'] & \boldsymbol{\alpha} \\ \boldsymbol{\beta} \arrow[r, "\Psi_\otimes(h)"] & \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \arrow[u, "\Psi_\otimes(g)"] \arrow[ur, "\Psi_\otimes(f)"'] & \end{tikzcd}$$

4. $0 \underset{\mathsf{PAL}^{\otimes}}{\longrightarrow} 0$

$$\begin{tikzcd} * \arrow[d, "1_*"'] \arrow[dr, "1_*"] & \\ * \arrow[r, "1_*"'] & * \\ 0 \arrow[r, "1_0"] & 0 \\ 0 \arrow[u, "1_0"] \arrow[ur, "1_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \alpha \arrow[d, "f^{-1}"'] \arrow[dr, "f^{-1}"] & \\ * \arrow[r, "1_*"'] & * \\ 0 \arrow[r, "1_0"] & 0 \\ \boldsymbol{\alpha} \arrow[u, "\Psi_\otimes(f^{-1})"] \arrow[ur, "\Psi_\otimes(f^{-1})"'] & \end{tikzcd}$$

The third case also uses the strategy applied in the first and the second ones. On the right side, we use the fact that $\Psi_\otimes$ is a functor.

The fourth case requires a proposition isomorphic to 0 and an action isomorphic to $*$, using the same strategy.

Definition 14. $\mathcal{PAL}^{\otimes}_{\bigstar} \xrightarrow{\Delta_\otimes} \mathcal{CNR}_{\bigstar}$ is a fibration defined by:

$$\Delta_\otimes(0) = 0$$

$$\Delta_\otimes(\boldsymbol{\alpha}) = O\alpha$$

1. $0 \underset{\mathsf{CNR}}{\longrightarrow} O\alpha$

$$\begin{tikzcd} 0 \arrow[d, "1_0"'] \arrow[dr, "f"] & \\ 0 \arrow[r, "f"'] & \boldsymbol{\alpha} \\ 0 \arrow[r, "\Delta_\otimes(f)"] & O\alpha \\ 0 \arrow[u, "1_0"] \arrow[ur, "\Delta_\otimes(f)"'] & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\beta} \arrow[d, "g^{-1}"'] \arrow[dr, "f\circ g^{-1}"] & \\ 0 \arrow[r, "f"'] & \boldsymbol{\alpha} \\ 0 \arrow[r, "\Delta_\otimes(f)"] & O\alpha \\ O\beta \arrow[u, "\Delta_\otimes(g^{-1})"] \arrow[ur, "\Delta_\otimes(f\circ g^{-1})"'] & \end{tikzcd}$$

2. $O\beta \underset{\mathsf{CNR}}{\longrightarrow} O\alpha$

$$\begin{tikzcd} 0 \arrow[d, "g"'] \arrow[dr, "f"] & \\ \boldsymbol{\beta} \arrow[r, "g\circ f^{-1}"'] & \boldsymbol{\alpha} \\ O\beta \arrow[r, "\Delta_\otimes(g\circ f^{-1})"] & O\alpha \\ 0 \arrow[u, "\Delta_\otimes(g)"] \arrow[ur, "\Delta_\otimes(f^{-1})"'] & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\gamma} \arrow[d, "h"'] \arrow[dr, "g"] & \\ \boldsymbol{\beta} \arrow[r, "f"'] & \boldsymbol{\alpha} \\ O\beta \arrow[r, "\Delta_\otimes(f)"] & O\alpha \\ O\gamma \arrow[u, "\Delta_\otimes(h)"] \arrow[ur, "\Delta_\otimes(g)"'] & \end{tikzcd}$$

3. $O\alpha \underset{\mathsf{CNR}}{\longrightarrow} 0$

$$\begin{tikzcd} 0 \arrow[d, "f"'] \arrow[dr, "1_0"] & \\ \boldsymbol{\alpha} \arrow[r, "f^{-1}"'] & 0 \\ O\alpha \arrow[r, "\Delta_\otimes(f^{-1})"] & 0 \\ 0 \arrow[u, "\Delta_\otimes(f)"] \arrow[ur, "1_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\beta} \arrow[d, "f\circ g^{-1}"'] \arrow[dr, "g^{-1}"] & \\ \boldsymbol{\alpha} \arrow[r, "f^{-1}"'] & 0 \\ O\alpha \arrow[r, "\Delta_\otimes(f^{-1})"] & 0 \\ O\beta \arrow[u, "\Delta_\otimes(f\circ g^{-1})"] \arrow[ur, "\Delta_\otimes(g^{-1})"'] & \end{tikzcd}$$

4. $0 \underset{\mathsf{CNR}}{\longrightarrow} 0$

$$\begin{tikzcd} 0 \arrow[d, "1_0"'] \arrow[dr, "1_0"] & \\ 0 \arrow[r, "1_0"'] & 0 \\ 0 \arrow[r, "1_0"] & 0 \\ 0 \arrow[u, "1_0"] \arrow[ur, "1_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\alpha} \arrow[d, "f^{-1}"'] \arrow[dr, "f^{-1}"] & \\ 0 \arrow[r, "1_0"'] & 0 \\ 0 \arrow[r, "1_0"] & 0 \\ O\alpha \arrow[u, "\Delta_\otimes(f^{-1})"] \arrow[ur, "\Delta_\otimes(f^{-1})"'] & \end{tikzcd}$$

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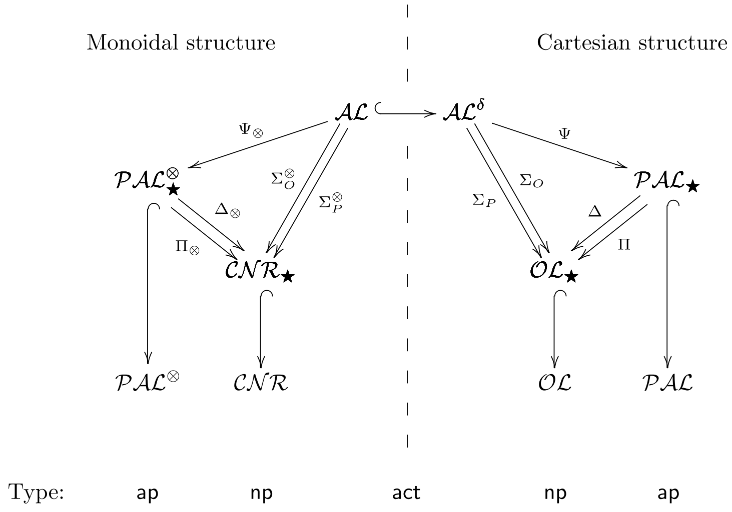

Fig. 1. $\mathcal{DDS}$.

As in the previous definition, we put f^{-1} such that $f^{-1} \circ f = 1_*$ and $f \circ f^{-1} = 1_\alpha$, similarly for g^{-1}, when there is an arrow from 0 to $\boldsymbol{\alpha}$ (resp. to $\boldsymbol{\beta}$). We require that $\Delta_\otimes(f^{-1}) = \Psi_\otimes(f)^{-1}$, and similarly for $\Delta_\otimes(g^{-1})$.

Definition 15. $\mathcal{PAL}^\otimes_\star \xrightarrow{\Pi_\otimes} \mathcal{CNR}_\star$ is a fibration defined similarly to $\Delta_\otimes$ (where P_s replaces O).

From these two fibrations we can also define two substitution rules such that $\Sigma^\otimes_O = \Delta_\otimes \circ \Psi_\otimes$ and $\Sigma^\otimes_P = \Pi_\otimes \circ \Psi_\otimes$ are fibrations. These definitions allow for substitution in $\mathcal{PAL}$ and $\mathcal{OL}$ (resp. $\mathcal{PAL}^\otimes$ and $\mathcal{CNR}$). Defining the rules as fibrations enables us to import and preserve the structure of $\mathcal{AL}$ into the other deductive systems.

The structure of $\mathcal{DDS}$ is represented in Fig. 1. On the left side of the dashed line, the deductive systems have a monoidal structure, as opposed to the Cartesian structure of the deductive systems on the right side.

Thus defined, $\mathcal{DDS}$ has an interesting property: actions that are isomorphic to $*$ are necessarily *false* in $\mathcal{PAL}$. This is accurate from a philosophical perspective since it would be impossible to accomplish both an action and its complement at the same time.

$$\Psi\big(\alpha \bullet \alpha^* \underset{\mathsf{AL}}{\longrightarrow} *\big) = \boldsymbol{\alpha} \bullet \boldsymbol{\alpha}^* \underset{\mathsf{PAL}}{\longrightarrow} \bot$$

The definition of $\mathcal{DDS}$ as a fibration allows us to preserve the relations between actions and obligations throughout $\mathcal{AL}$, $\mathcal{PAL}$ and $\mathcal{OL}$ (resp. $\mathcal{AL}$, $\mathcal{PAL}^\otimes$ and $\mathcal{CNR}$). Hence, if two actions are related (by some arrow) in $\mathcal{AL}$, then they will also be linked in $\mathcal{PAL}$ and $\mathcal{OL}$ (resp. $\mathcal{PAL}^\otimes$ and $\mathcal{CNR}$). It also allows us to distinguish between actions that are impossible to perform versus action propositions that are necessarily false. This answers the philosophical intuition that the notions of *tautologous* and *contradictory* actions do not make any sense. Although a conjunctive action might be impossible to perform, the actions are not *contradictory*: they are only incompatible. That said, if it is impossible to accomplish α and β together, then necessarily $\boldsymbol{\alpha} \bullet \boldsymbol{\beta}$ will be false. This is exactly what we obtain through the definition of (Ψ).

$$\Psi(\alpha \bullet \beta \underset{\mathsf{AL}}{\longrightarrow} *) = \boldsymbol{\alpha} \bullet \boldsymbol{\beta} \underset{\mathsf{PAL}}{\longrightarrow} \bot$$

Combined with (Δ), this entails an interesting property: actions isomorphic to $*$ cannot be obligatory. The same results hold for $\mathcal{PAL}^\otimes$ and $\mathcal{CNR}$.

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$$\Delta(\boldsymbol{\alpha} \bullet \boldsymbol{\alpha}^* \underset{\mathsf{PAL}}{\longrightarrow} \bot) = O(\alpha \bullet \alpha^*) \underset{\mathsf{OL}}{\longrightarrow} \bot$$
$$\Delta(\boldsymbol{\alpha} \bullet \boldsymbol{\beta} \underset{\mathsf{PAL}}{\longrightarrow} \bot) = O(\alpha \bullet \beta) \underset{\mathsf{OL}}{\longrightarrow} \bot$$

Looking at (Σ_O), this allows us to formally represent the *ought implies can* principle: if it is impossible to accomplish some action α (i.e., $\alpha \cong *$), then it is false that α is obligatory. As such, actions that are isomorphic to the action which is impossible to perform cannot be obligatory. Hence, 'ought' implies 'can'. An interesting feature of $\mathcal{DDS}$ is that it does not require any alethic modality to represent this principle.

To strengthen the ought implies can principle, one could add the axiom $(\Diamond)$ to $\mathcal{DDS}$, although it would not add much significance to the ought implies can principle.[26]

$$\frac{}{O * \underset{\mathsf{OL}}{\longrightarrow} \bot} \qquad (\Diamond)$$

5. Residuated monoids

From a semantical point of view, each type of deductive system has its algebraic counterpart. It is well-known that what we might call *monoidal logics* are sound and complete with respect to different variations of partially ordered residuated monoids (cf. [92]) and moreover that intuitionistic and classical logics are sound and complete with respect to distributive lattices and complemented distributive lattices. Given that $\mathcal{PAL}$ and $\mathcal{OL}$ are two variations of intuitionistic and classical logics, and thus that their completeness proofs are trivial, we will only provide proofs for the other fragments of $\mathcal{DDS}$. In what follows, we will define the semantics for each fragment, and the overall semantics will result from the appropriate fibrations between these algebraic structures.

There are currently various sound and complete semantics for the propositional multiplicative fragment of linear logic **MLL**, and also for other fragments such as **MALL** or **MALL** + MIX. It is actually a well-known result that models for **MALL** are $*$-autonomous categories with products and co-products, while models for **MLL** are simply $*$-autonomous categories [106,11]. Abramsky and Jagadeesan [1] were the first to introduce a proper categorical (game) semantics for **MLL** (+ MIX), and since then there as been numerous proposals (see [16] for the history and evolution of categorical semantics for fragments of linear logic).[27]

One consequence of the categorical understanding of logic is that it requires that we think of a semantics for *proofs* rather than a semantics for *provability* [1,17]. This implies a subtle shift of perspective: the logic must not be thought of as a collection of *theorems* (propositions) but must rather be understood as a collection of *proofs* (arrows). As such, the aim is not to determine whether or not a *proposition* is derivable from a collection of propositions, but it is rather to determine if a *proof* can be derived from a collection of proofs.

Having this in mind, the objective is not to determine whether or not a *formula* is valid, but it is rather to establish whether or not a *proof* is valid. Hence, the notion of validity for proofs must be defined. Following Blute and Scott [17], when a logic is understood as a (free) category $\mathcal{C}$, a model $\mathcal{M}$ can be defined via a (free) functor $v : \mathcal{C} \longrightarrow \mathcal{M}$. The *soundness* of $\mathcal{C}$ is trivially obtained by defining v as a functor. As such, we automatically have that if there is a proof f in $\mathcal{C}$, then there is a semantical consequence $v(f)$ in $\mathcal{M}$. *Full completeness* is obtained when the valuation functor is *full*, that is, when any $g : v(\varphi) \longrightarrow v(\psi)$ in $\mathcal{M}$ is $v(f)$ for some $f : \varphi \longrightarrow \psi$ in $\mathcal{C}$ (cf. [1]). Thus, full completeness yields that for every semantical relation $v(f)$ in $\mathcal{M}$ there is a proof f in $\mathcal{C}$.[28]

We follow the algebraic framework of [48] for residuated and involutive monoids. We adopt the following definitions. Let $\langle \boldsymbol{M}, \leq, \cdot, 1, 0\rangle$ be a partially ordered monoid (hereafter po-monoid) with a special object 0.

[26] Note that $\Delta(*) = O*$.
[27] Other semantics were proposed for similar systems (e.g., [7,68]).
[28] One could go further and define v as a fully *faithful* functor, in which case if $v(f) = v(g)$, then $f = g$.

1. $\mathsf{M} = \langle \boldsymbol{M}, \leq, \cdot, 1, \backslash \rangle$ is *residuated* when:

$$p \cdot q \leq r \quad \text{iff} \quad p \leq q \backslash r$$

2. $\mathsf{M} = \langle \boldsymbol{M}, \leq, \cdot, 1, 0, \backslash \rangle$ is *involutive* when:

$$(p \backslash 0) \backslash 0 = p$$

3. $\mathsf{M} = \langle \boldsymbol{M}, \leq, \cdot, 1, \backslash \rangle$ is *commutative* when:

$$p \cdot q = q \cdot p$$

4. $\mathsf{M} = \langle \boldsymbol{M}, \leq, \cdot, 1, \backslash \rangle$ is *compact* when it is residuated, commutative and (here, note that $0 = 1$):

$$(p \backslash 1) \cdot q = p \backslash q$$

Definition 16. An arrow $f : \varphi \longrightarrow \psi$ is *valid* if and only if $v(\varphi) \leq v(\psi)$ for all valuation v.

Let $\mathbb{D}$ be a deductive system and $v : \mathbb{D} \longrightarrow \mathsf{M}$ a valuation functor defined by:

$$\begin{aligned} v(0) &= 0 \\ v(1) &= 1 \\ v(\varphi \otimes \psi) &= v(\varphi) \cdot v(\psi) \\ v(\varphi \multimap \psi) &= v(\varphi) \backslash v(\psi) \\ v(\varphi \parr \psi) &= v(\varphi) + v(\psi) \end{aligned}$$

We use different valuations v for each deductive system. It is well-known that classical and intuitionistic logic are respectively sound and complete with Boolean and Heyting algebras (see for instance [52]). As such, we can easily prove the following results.

Theorem 1. *$\mathcal{PAL}$ is sound and complete with respect to distributive lattices.*

Theorem 2. *$\mathcal{OL}$ is sound and complete with respect to distributive complemented lattices satisfying:*

$$\begin{aligned} v(O\alpha) &\leq v(P_s\alpha) \\ v(P_s\alpha) &\leq v(P_w\alpha) \end{aligned}$$

Given the results presented in [92], we also know that:

Theorem 3. *$\mathcal{PAL}^{\otimes}$ is sound and complete with respect to commutative residuated po-monoids.*

As such, it remains to show that $\mathcal{AL}$ and $\mathcal{CNR}$ possess a sound and complete interpretation. It is a well-known result that the algebraic counterpart of **MALL** is a residuated lattice [29–31,69,87,48] (or, in the terminology of [6], a consequence algebra) and, incidentally, that the algebraic counterpart of **MLL** is a partially ordered residuated commutative monoid (or, in the terminology of [39], a commutative Lambek monoid).

Theorem 4. *$\mathcal{CNR}$ is sound and complete with respect to commutative involutive residuated po-monoids satisfying:*

$$v(O\alpha) \leq v(P_s\alpha)$$
$$v(P_s\alpha) \leq v(P_w\alpha)$$

Proof. It follows from Theorems 5 and 6. □

Theorem 5 *(Soundness). If $f : \varphi \longrightarrow \psi$ is a $\mathcal{CNR}$-arrow, then f is valid.*

Proof. We show that the rules and axioms of $\mathcal{CNR}$ preserve validity in M.

(1) $v(\varphi) = v(\varphi)$, hence $v(\varphi) \leq v(\varphi)$.
($\sim\sim$) $v(\sim\sim\varphi) = v(\sim\varphi)\backslash 0 = (v(\varphi)\backslash 0)\backslash 0 = v(\varphi)$.
(t) It follows from the fact that M satisfies increasing monotony: assume that $v(\varphi) \leq v(\psi)$ and $v(\rho) \leq v(\tau)$, then $v(\varphi) \cdot v(\rho) \leq v(\psi) \cdot v(\tau)$.

$$v(\psi) \cdot v(\tau) \leq v(\psi) \cdot v(\tau)$$
$$v(\tau) \leq v(\psi)\backslash(v(\psi) \cdot v(\tau))$$
$$v(\rho) \leq v(\psi)\backslash(v(\psi) \cdot v(\tau))$$
$$v(\rho) \cdot v(\psi) \leq v(\psi) \cdot v(\tau)$$
$$v(\psi) \cdot v(\rho) \leq v(\psi) \cdot v(\tau)$$
$$v(\psi) \leq v(\rho)\backslash(v(\psi) \cdot v(\tau))$$
$$v(\varphi) \leq v(\rho)\backslash(v(\psi) \cdot v(\tau))$$
$$v(\varphi) \cdot v(\rho) \leq v(\psi) \cdot v(\tau)$$

(cut) Assume that $v(\varphi) \leq v(\psi)$ and $v(\psi) \leq v(\rho)$. Hence, $v(\varphi) \leq v(\rho)$.
(cl) ($\Downarrow$) Assume $v(\varphi \otimes \psi) \leq v(\rho)$. Hence, $v(\varphi) \cdot v(\psi) \leq v(\rho)$, and it follows by residuation that $v(\varphi) \leq v(\psi)\backslash v(\rho)$, thus $v(\varphi) \leq v(\psi \multimap \rho)$. ($\Uparrow$) Assume $v(\varphi) \leq v(\psi \multimap \rho)$, hence $v(\varphi) \leq v(\psi)\backslash v(\rho)$ and by residuation $v(\varphi) \cdot v(\psi) \leq v(\rho)$, therefore $v(\varphi \otimes \psi) \leq v(\rho)$.
(b) Since M is a commutative monoid, $v(\varphi \otimes \psi) = v(\varphi) \cdot v(\psi) = v(\psi) \cdot v(\varphi) = v(\psi \otimes \varphi)$.
(r) ($\Downarrow$) Assume $v(\varphi) \leq v(\psi \otimes 1)$. Hence, $v(\varphi) \leq v(\psi) \cdot v(1)$, and since M is a monoid, $v(\psi) \cdot v(1) = v(\psi) \cdot 1 = v(\psi)$, thus $v(\varphi) \leq v(\psi)$. ($\Uparrow$) Assume $v(\varphi) \leq v(\psi)$. Since M is a monoid we have $v(\psi) \cdot v(1) = v(\psi) \cdot 1 = v(\psi)$, hence $v(\varphi) \leq v(\psi) \cdot v(1)$ and therefore $v(\varphi) \leq v(\psi \otimes 1)$.
(a) M is a monoid, hence $v(\varphi \otimes (\psi \otimes \rho)) = v(\varphi) \cdot v(\psi \otimes \rho) = v(\varphi) \cdot (v(\psi) \cdot v(\rho)) = (v(\varphi) \cdot v(\psi)) \cdot v(\rho) = v(\varphi \otimes \psi) \cdot v(\rho) = v((\varphi \otimes \psi) \otimes \rho)$.
(**D**) Trivial.
(**P**) Trivial. □

By construction, it is easy to show that v is a functor: it is defined for every objects and arrows of $\mathcal{CNR}$ and it preserves identities and compositions, as shown in the commutative diagrams below.

$$\begin{tikzcd}
v(\varphi) \arrow[d, "\overset{v(f)}{\leq}"'] \arrow[dr, "\overset{v(gf)=v(g)v(f)}{\leq}"] & \\
v(\psi) \arrow[r, "\overset{v(g)}{\leq}"'] & v(\rho)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v(\psi) \arrow[d, "\overset{v(1_\psi)}{\leq}"] \arrow[dr, "\overset{v(g)v(1_\psi)=v(g)}{\leq}"] & \\
v(\varphi) \arrow[ur, "\overset{v(f)}{\leq}"] \arrow[r, "\overset{v(1\psi)v(f)=v(f)}{\leq}"'] & v(\psi) \arrow[r, "\overset{v(g)}{\leq}"'] & v(\rho)
\end{tikzcd}$$

It thus remains to show that $\mathcal{CNR}$ is fully complete by showing that v is full. As such, we need to show that if there is a valid $g \in \mathsf{M}$, then there is $f \in \mathcal{CNR}$ such that $g = v(f)$. To do so, we follow [92] and adapt the well-known method of maximally consistent sets to the categorical (arrow-theoretical) understanding of logic. This requires that we introduce the notion of consistency between *proofs*.

The categorical understanding of logic requires a shift of perspective. In the usual settings, we would say that a logic is inconsistent when 0 is in the system. However, this conception cannot simply be imported to deductive systems: we cannot say that a deductive system is inconsistent when 0 is in it since by definition it is! Similarly, we would say that two propositions φ and ψ are inconsistent if together they lead to 0. But even though two propositions are inconsistent, $\varphi \otimes \psi \longrightarrow 0$ remains an acceptable proof.

Before presenting a syntactical notion of consistency between proofs, let us see first what it would mean for two arrows to be inconsistent from a semantical point of view. It is noteworthy that allowing the derivation of $1 \longrightarrow 0$ does not suffice to conclude that f and g are inconsistent within a model. Indeed, 1 is not a greatest element of M (nor is 0 the least), and as such $0 \leq 1$ is not valid. Hence, it is possible to have models where $1 \leq 0$. This might seem counter intuitive at first glance, but it is simply a consequence of being within the monoidal (or linear) world. There would be a problem, however, if we were in a situation such that $v(\varphi) = v(\sim \varphi)$ or $0 = 1$. In these cases, M would not be a model of $\mathcal{CNR}$.

Having this understanding in mind, we can now define the notion of consistency between arrows. In a nutshell, the idea is to say that a collection of arrows is *inconsistent* when it allows an isomorphism between a formula and its negation. Given two formulas φ and ψ, consider the eight possible combinations, which by contraposition can be reduced to four arrows:

$$\begin{array}{cc} \varphi \xrightarrow{f} \psi & \varphi \xrightarrow{\dot{f}} \sim \psi \\ \sim \varphi \xrightarrow{f^{\sim}} \psi & \psi \xrightarrow{f^{-1}} \varphi \end{array}$$

Definition 17. $\mathbb{C}$ is *inconsistent* only if either (i) there are φ and ψ such that f, $\dot{f}$, $f^{\sim}$ and f^{-1} are $\mathbb{C}$-arrows, or (ii) both $0 \longrightarrow 1$ and $1 \longrightarrow 0$ are $\mathbb{C}$-arrows.

Lemma 1. *If $\mathbb{C}$ is inconsistent, then either $v(\varphi) = v(\sim \varphi) = v(\psi) = v(\sim \psi)$ or $0 = 1$.*

Proof. Assume that f, $\dot{f}$, $f^{\sim}$ and f^{-1} are $\mathbb{C}$-arrows. Hence, we have:

$$\begin{array}{rl} v(\varphi) \leq_1 v(\psi) & v(\varphi) \leq_2 v(\sim \psi) \\ v(\sim \varphi) \leq_3 v(\psi) & v(\psi) \leq_4 v(\varphi) \end{array}$$

We can easily show that $v(\varphi) \leq v(\psi)$ if and only if $v(\sim \psi) \leq v(\sim \varphi)$ (the proof is left to the reader). From our assumptions, we thus obtain:

$$v(\varphi) \leq_1 v(\psi) \leq_2 v(\sim \varphi) \leq_3 v(\psi) \leq_4 v(\varphi) \leq_2 v(\sim \psi) \leq_3 v(\varphi) \leq_1 v(\psi)$$

As such, we have:

$$\begin{array}{ll} v(\varphi) \leq v(\sim \varphi) & v(\sim \varphi) \leq v(\varphi) \\ v(\psi) \leq v(\sim \psi) & v(\sim \psi) \leq v(\psi) \end{array}$$

Therefore $v(\varphi) = v(\sim \varphi) = v(\psi) = v(\sim \psi)$. On the other hand, if there are $1 \longrightarrow 0$ and $0 \longrightarrow 1$, then $v(1) \leq v(0)$ and $v(0) \leq v(1)$, hence $0 = 1$. □

Let $\langle \varphi, \psi \rangle_1, \ldots, \langle \rho, \tau \rangle_n, \ldots$ be an arbitrary enumeration of all pairs of objects of $\mathbb{C}_1 = \mathcal{CNR}$. We use the notation $\mathbb{C}_i, f_i$ to refer to the addition of f_i to $\mathbb{C}_i$ (i.e., adding f_i as an axiom). Each $\mathbb{C}_{i+1}$ is constructed

in four steps. The idea is to construct a $\mathbb{C}_{i+1}$ that contains the maximum number of the following arrows without breaking the consistency: f, $\dot{f}$, $f^{\sim}$ and f^{-1}. Here is the procedure to construct each $\mathbb{C}_{i+1}$:

1. If $\mathbb{C}_i$, f_i is inconsistent, then $\mathbb{C}_{i+1} = \mathbb{C}_i$, otherwise $\mathbb{C}_i^1 = \mathbb{C}_i, f_i$.
2. If $\mathbb{C}_i^1$, $\dot{f}_i$ is inconsistent, then $\mathbb{C}_{i+1} = \mathbb{C}_i^1$, otherwise $\mathbb{C}_i^2 = \mathbb{C}_i^1, \dot{f}_i$.
3. If $\mathbb{C}_i^2$, $f_i^{\sim}$ is inconsistent, then $\mathbb{C}_{i+1} = \mathbb{C}_i^2$, otherwise $\mathbb{C}_{i+1} = \mathbb{C}_i^2, f_i^{\sim}$.

Now, consider an extension $\mathbb{C}_\omega$ such that:

$$\mathbb{C}_\omega = \bigcup_{i=1}^{\infty} \mathbb{C}_i$$

Lemma 2. $\mathbb{C}_1$ *is consistent.*

Proof. It follows from Theorem 5 and Lemma 1: if $\mathbb{C}_1$ was inconsistent, then either (i) there is some f such that f, $\dot{f}$, $f^{\sim}$ and f^{-1} are $\mathbb{C}_1$-arrows, hence for each v we would have $v(\varphi) = v(\sim \varphi) = v(\psi) = v(\sim \psi)$, or (ii) there are $0 \longrightarrow 1$ and $1 \longrightarrow 0$, and hence in both cases there would not be any model of $\mathbb{C}_1$. □

Lemma 3. *If* $\mathbb{C}_i$ *is consistent, then so is* $\mathbb{C}_{i+1}$.

Proof. We proceed by induction. Assume that $\mathbb{C}_{i+1}$ is inconsistent. This implies that the addition of some f during its construction broke its consistency. However, this is impossible:

1. if $\mathbb{C}_i$, f is inconsistent, we have $\mathbb{C}_{i+1} = \mathbb{C}_i$, hence f is not in $\mathbb{C}_{i+1}$ and hence it is consistent, otherwise $\mathbb{C}_i^1 = \mathbb{C}_i, f$;
2. if $\mathbb{C}_i^1$, $\dot{f}$ is inconsistent, we have $\mathbb{C}_i^1 = \mathbb{C}_{i+1}$, hence $\dot{f}$ is not in $\mathbb{C}_{i+1}$ and by (HI) it is consistent, otherwise $\mathbb{C}_i^2 = \mathbb{C}_i, f, \dot{f}$;
3. if $\mathbb{C}_i^2$, $f^{\sim}$ is inconsistent, we have $\mathbb{C}_i^2 = \mathbb{C}_{i+1}$, hence $f^{\sim}$ is not in $\mathbb{C}_{i+1}$ and as such it is consistent, otherwise $\mathbb{C}_{i+1} = \mathbb{C}_i, f, \dot{f}, f^{\sim}$, and hence f^{-1} is not in $\mathbb{C}_i$, meaning that $\mathbb{C}_i, f, \dot{f}, f^{\sim}$ is consistent. □

Lemma 4. *Each* $\mathbb{C}_\omega$ *is consistent.*

Proof. Assume that there is an inconsistent $\mathbb{C}_\omega$. This implies that there is an inconsistent $\mathbb{C}_{i+1}$, which is impossible according to the previous lemma. □

By construction, it is quite trivial that $\mathbb{C}_\omega$ is an extension of $\mathbb{C}_1$. As such, if f is a $\mathbb{C}_1$-arrow, then it is also a $\mathbb{C}_\omega$-arrow for all $\mathbb{C}_\omega$. The converse is also actually true, as it is shown in Lemma 7.

By definition of consistency, we have the following corollary. Consider the case for $\mathbb{C} = \mathbb{C}_i, \dot{f}, f^{\sim}, f^{-1}$.

Corollary 1. *If* f *is not a* $\mathbb{C}_i$*-arrow and either* $1 \longrightarrow 0$ *or* $0 \longrightarrow 1$ *is not an arrow of* $\mathbb{C}_i, \dot{f}, f^{\sim}, f^{-1}$*, then* $\mathbb{C}, \dot{f}, f^{\sim}, f^{-1}$ *is consistent.*

Lemma 5. *If* f *is not a* $\mathbb{C}_1$*-arrow, then either* $1 \longrightarrow 0$ *is not an arrow of* $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$ *or* $0 \longrightarrow 1$ *is not an arrow of* $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$.

Proof. Assume that f is not a $\mathbb{C}_1$-arrow, but that both $1 \longrightarrow 0$ and $0 \longrightarrow 1$ are arrows of $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$. We know that neither $1 \longrightarrow 0$ nor $0 \longrightarrow 1$ are $\mathbb{C}_1$-arrows. As such, we only need to show that it is impossible to have both $1 \longrightarrow 0$ and $0 \longrightarrow 1$ in $\dot{f}$, $f^{\sim}$, f^{-1}.

1. Consider first the possibility that $1 \longrightarrow 0$ reduces to $\dot{f}$, $f^{\sim}$ or f^{-1}.
   (a) If it reduces to an equivalent arrow in a $\dot{f}$-form, then $\dot{f} :\sim 0 \longrightarrow \sim 1$, hence $f :\sim 0 \longrightarrow 1$. By contraposition, this arrow is equivalent to the arrow $\sim 1 \longrightarrow 0$, which is in $\mathbb{C}_1$ (cf. Appendix B) and hence contradicts our first assumption.
   (b) If it reduces to an equivalent arrow in an $f^{\sim}$-form, then $f^{\sim} :\sim 0 \longrightarrow \sim 1$, hence $f : 0 \longrightarrow \sim 1$. However, this arrow is derivable within $\mathbb{C}_1$ (cf. Appendix B), which contradicts our first hypothesis.
   (c) If it reduces to an equivalent arrow in an f^{-1}-form, then $f^{-1} : 1 \longrightarrow 0$, hence $f : 0 \longrightarrow 1$. Hence, f is neither in $\mathbb{C}_1$ nor in $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$.
2. Secondly, consider the possibility that $0 \longrightarrow 1$ reduces to $\dot{f}$, $f^{\sim}$ or f^{-1}.
   (a) If it reduces to an equivalent arrow in a $\dot{f}$-form, then $\dot{f} :\sim 1 \longrightarrow \sim 0$, hence $f :\sim 1 \longrightarrow 0$, but this is a $\mathbb{C}_1$-arrow.
   (b) If it reduces to an equivalent arrow in an $f^{\sim}$-form, then $f^{\sim} :\sim 1 \longrightarrow \sim 0$, hence $f : 1 \longrightarrow \sim 0$, which is also a $\mathbb{C}_1$-arrow.
   (c) If it reduces to an equivalent arrow in an f^{-1}-form, then $f^{-1} : 0 \longrightarrow 1$, hence $f : 1 \longrightarrow 0$. Hence, f is neither in $\mathbb{C}_1$ nor in $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$. □

Lemma 6. *If f is not a $\mathbb{C}_1$-arrow, then $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$ is consistent.*

Proof. It follows from Lemma 5 and Corollary 1. □

Lemma 7. *If f is a $\mathbb{C}_\omega$-arrow for all $\mathbb{C}_\omega$, then f is a $\mathbb{C}_1$-arrow.*

Proof. Assume that f is a $\mathbb{C}_\omega$-arrow for all $\mathbb{C}_\omega$ but is not a $\mathbb{C}_1$-arrow. By Lemma 6, this implies that $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$ is consistent, hence it is a subcategory for some $\mathbb{C}_\omega$, and thus f is not in all $\mathbb{C}_\omega$, otherwise it would be inconsistent, contradicting Lemma 4. □

Lemma 8. M *is a model of* $\mathbb{C}_\omega$.

Proof. This follows from the definition of $v : \mathbb{C}_\omega \longrightarrow \mathsf{M}$ as a functor. □

Theorem 6 *(Full completeness). If $v(\varphi) \leq v(\psi)$ for all valuations, then there is a $\mathbb{C}_1$-arrow $f : \varphi \longrightarrow \psi$ such that $v(f) = v(\varphi) \leq v(\psi)$.*

Proof. Assume that $v(\varphi) \leq v(\psi)$ is valid, but that for every $\mathbb{C}_1$-arrow f, either f is not in $\mathbb{C}_1$ or f is not mapped to $v(\varphi) \leq v(\psi)$.

1. If f is not in $\mathbb{C}_1$, then, by Lemma 6, $\mathbb{C}_1, \dot{f}, f^{\sim}, f^{-1}$ is consistent. Hence, it possesses a maximally consistent extension $\mathbb{C}_\omega$, and by Lemma 8 we have:

$$v(\varphi) \leq v(\sim \psi)$$
$$v(\sim \varphi) \leq v(\psi)$$
$$v(\psi) \leq v(\varphi)$$

   By the proof of Lemma 1, we can show that $v(\varphi) = v(\sim \varphi) = v(\psi) = v(\sim \psi)$, hence that M is not a model of $\mathbb{C}_\omega$, contradicting Lemma 8.
2. If f is not mapped to $v(\varphi) \leq v(\psi)$, then either f is in $\mathbb{C}_1$ or it is not. If it is not, the aforementioned reasoning applies. If it is, then by Theorem 5 f is mapped to $v(\varphi) \leq v(\psi)$. □

This complete the proof for $\mathcal{CNR}$. Let us note that, building on Baez's and Stay's [9] work, we showed in [91] that $\mathcal{CNR}$ is not only sound and complete with respect to string diagrams, but that it is also decidable. Also, note that a string diagrammatic semantics could have been provided for $\mathcal{DDS}$ instead of an algebraic one.

It thus remains to show that $\mathcal{AL}$ possesses a sound and complete interpretation. An *action algebra* $\mathsf{A} = \langle \mathbf{A}, \leq, \cdot, \backslash, *, \circ \rangle$ is defined on the grounds of a po-monoid $\langle \mathbf{A}, \leq, \circ, * \rangle$ together with a compact po-monoid $\langle \mathbf{A}, \leq, \cdot, \backslash, * \rangle$. A valuation $v : \mathcal{AL} \longrightarrow \mathsf{A}$ is a functor defined by:

$$\begin{aligned} v(*) &= * \\ v(\alpha \bullet \beta) &= v(\alpha) \cdot v(\beta) \\ v(\alpha \curvearrowright \beta) &= v(\alpha) \circ v(\beta) \\ v(\beta \ominus \alpha) &= v(\alpha) \backslash v(\beta) \end{aligned}$$

Theorem 7 *(Soundness). If $f : \alpha \longrightarrow \beta$ is an $\mathcal{AL}$-arrow, then f is valid.*

Proof. Given Theorem 5, it trivially follows from the properties of an action algebra: we only need to show that $(\varphi \backslash *) \cdot \psi = \varphi \backslash \psi$. $\square$

$\mathcal{CNR}$'s completeness proof used the notion of maximally consistent collection of arrows. In a nutshell, the strategy is to show that if an arrow f is not in $\mathcal{CNR}$, then there is a maximally consistent collection of arrows that has a model M, but if f is valid, then this collection does not have any model since it implies that there is φ such that $v(\varphi) = v(\sim \varphi)$. This method, however, cannot directly be applied to a compact deductive system: there *are* models of $\mathcal{AL}$ satisfying the property that there is α such that $v(\alpha) = v(\alpha^*)$. For instance, $v(*) = v(*^*)$. Nonetheless, the strategy can be adapted: we will show that if an arrow is not in $\mathcal{AL}$, then there is a maximally compact-consistent extension $\mathbb{D}_\omega$ that has a model M with a specific property. However, if f is valid, then there will not be any model satisfying this property.

Given a deductive system $\mathbb{D}$, let $[\alpha]_{\mathbb{D}}$ stand for the class of equivalence where propositions are isomorphic to α in regards to $\mathbb{D}$. Let $eq : \mathcal{AL} \longrightarrow [\mathcal{AL}]$ be a functor that maps each proposition in $\mathcal{AL}$ to its equivalence class in $[\mathcal{AL}]$. We define $[v] : [\mathcal{AL}] \longrightarrow \mathsf{A}$ by:

$$\begin{aligned} [v]\big([*]\big) &= * \\ [v]\big([\alpha \bullet \beta]\big) &= [v]\big([\alpha]\big) \cdot [v]\big([\beta]\big) \\ [v]\big([\alpha \curvearrowright \beta]\big) &= [v]\big([\alpha]\big) \circ [v]\big([\beta]\big) \\ [v]\big([\beta \ominus \alpha]\big) &= [v]\big([\alpha]\big) \backslash [v]\big([\beta]\big) \end{aligned}$$

It is easy to show that $[v]$ is injective, hence that v is injective with respect to equivalence classes. Indeed, since the valuation for two isomorphic formulas is equal, we can show that $v(\alpha) = [v](eq(\alpha))$, and as such if $v(\alpha) = v(\beta)$, then $\alpha, \beta \in [\alpha]_{\mathbb{D}}$. Now, consider the following arrows:

$$\begin{array}{cc} \alpha \xrightarrow{f} \beta & \beta \xrightarrow{f^{-1}} \alpha \\ \alpha \xrightarrow{f_*} \beta^* & \beta^* \xrightarrow{f_*^{-1}} \alpha \end{array}$$

Considering that $* \cong *^*$, we adapt the previous definition of consistency.

Definition 18. $\mathbb{D}$ is *compact-inconsistent* if and only if there is $\alpha \ncong \alpha^*$ such that f, f^{-1}, f_* and f_*^{-1} are $\mathbb{D}$-arrows.

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Corollary 2. *If $f \notin \mathcal{AL}$, then $\mathcal{AL}, f^{-1}, f_*, f_*^{-1}$ is compact-consistent.*

Proof. Assume that $f \notin \mathcal{AL}$ but that the collection $\mathbb{D} = \mathcal{AL}, f^{-1}, f_*, f_*^{-1}$ is compact-inconsistent. It follows that there is $\alpha \ncong \alpha^*$ such that f, f^{-1}, f_* and f_*^{-1} are $\mathbb{D}$-arrows, hence $f \in \mathcal{AL}$. □

By adapting the aforementioned procedure, we can construct a maximally compact-consistent extension $\mathbb{D}_\omega$. Let $\mathbb{D}_1 = \mathcal{AL}$.

Lemma 9. $\mathbb{D}_1$ *is compact-consistent.*

Proof. Assume that $\mathbb{D}_1$ is compact-inconsistent. It follows that there is $\alpha \ncong \alpha^*$ such that f, f^{-1}, f_* and f_*^{-1} are $\mathbb{D}_1$-arrows. However, we can easily show from these arrows that $\alpha \cong \alpha^*$. □

Lemma 10. *If $f \notin \mathcal{AL}$, then there is a maximally compact-consistent extension $\mathbb{D}_\omega$ such that $\alpha \notin [\beta]_{\mathbb{D}_\omega}$.*

Proof. From the previous lemma, if $f \notin \mathcal{AL}$, then $\mathcal{AL}, f^{-1}, f_*, f_*^{-1}$ is compact-consistent, hence it possesses a maximally compact-consistent extension $\mathbb{D}_\omega$. However, if $\alpha \in [\beta]_{\mathbb{D}_\omega}$, then $f \in \mathbb{D}_\omega$, hence $\mathbb{D}_\omega$ is compact-inconsistent. □

Lemma 11. A *is a model of* $\mathbb{D}_\omega$*.*

Proof. It follows from the fact that $v : \mathbb{D}_\omega \longrightarrow \mathsf{A}$ is a functor. □

Theorem 8 *(Full completeness). If $v(\alpha) \leq v(\beta)$ for all valuations, then there is a $\mathbb{D}_1$-arrow $f : \alpha \longrightarrow \beta$ such that $v(f) = v(\alpha) \leq v(\beta)$.*

Proof. Assume that $v(\alpha) \leq v(\beta)$ is valid, but that for every $\mathbb{D}_1$-arrow f, either f is not in $\mathbb{D}_1$ or f is not mapped to $v(\alpha) \leq v(\beta)$.

1. If f is not in $\mathbb{D}_1$, then, by Corollary 2, $\mathbb{D}_1, f^{-1}, f_*, f_*^{-1}$ is compact-consistent. By Lemma 10, it follows that there is a maximally compact-consistent extension $\mathbb{D}_\omega$ such that $\alpha \notin [\beta]_{\mathbb{D}_\omega}$. However, if f is valid, then $v(\alpha) = v(\beta)$, and therefore $\alpha, \beta \in [\beta]_{\mathbb{D}_\omega}$ since v is injective with respect to equivalence classes.
2. If f is not mapped to $v(\alpha) \leq v(\beta)$, then either f is in $\mathbb{D}_1$ or it is not. If it is not, the aforementioned reasoning applies. If it is, then by Theorem 7 f is mapped to $v(\alpha) \leq v(\beta)$. □

Summing up, each fragment of $\mathcal{DDS}$ is sound and complete with respect to its algebraic counterpart within residuated po-monoids. The overall semantics can be obtained by defining the appropriate fibrations between the models.

To the category theorist, the completeness proof will perhaps appear as a trivial result. After all, a completeness proof amounts to show that two structures can be embedded into another. Once we know that a deductive system $\mathbb{D}$ and an algebra A are both instances of a category $\mathcal{C}$, it becomes easy to construct a structure preserving functor from $\mathbb{D}$ to A.

Although we agree that looking at logic from a categorical perspective facilitates the construction of a sound and complete interpretation, we do not think, however, that the completeness result should be seen as trivial. So far, the reader might have wondered what are the benefits of adopting category theory as a foundation for deontic logic. As it happens, this relation between syntax and semantics is actually an advantage of understanding logic categorically. By defining a deductive system as a specific instance of a (free) category, we directly obtain the structure of the semantics we need.

The upshot of the categorical understanding of logic is that when the syntax is defined properly, one obtains directly the semantics via the categorical structure of the deductive system. In this respect, one can understand why it is often said that a categorical approach to logic loosens up the usual dichotomy between syntax and semantics. As deductive systems can be defined as instances of free categories, different algebras can be seen in the same light. For instance, a partially ordered set can be seen as a category while a partially ordered monoid can be seen as a monoidal category. Similarly, a residuated partially ordered monoid is an instance of a monoidal closed category and a commutative partially ordered monoid can be seen as a symmetric monoidal category. As such, from a categorical perspective, it becomes quite trivial that residuated lattices (cf. [48]) offer a semantics for various substructural (or monoidal) logics given that many of these logics can be defined as deductive systems. Understanding logic from a categorical point of view thus enables us to easily obtain an algebraic semantics for our deductive systems.

6. A comparison with Goble's analysis

Goble [51] provided a thorough analysis of the conditions that a logic which aims to model normative conflicts should satisfy.[29] His analysis is grounded on two fundamental arguments [51, pp. 450–451].

Argument 1. Aggregation of obligations, together with the ought implies can principle, leads to the impossibility of normative conflicts.

Argument 2. The distribution principle 'if necessarily α implies β, then $O\alpha$ implies $O\beta$', together with the axiom schema for normative consistency, leads to the impossibility of normative conflicts.

The four principles at play within these arguments are:

(Agg) If α is obligatory and β is obligatory, then their conjunction is also obligatory.
(Can) If α is obligatory, then it is possible to accomplish α.
(Dist) If α implies β, then if α is obligatory, then β is obligatory.
(D) If α is obligatory, then not-α is not.

Despite the second argument, Goble [51, p. 469] argues that (Dist) must not be rejected altogether given that it is a fundamental principle for a logic of ought. We agree. It enables us to infer from a set of ought statements other oughts that are not explicitly mentioned themselves. In this respect, a form of distribution is necessary to minimally represent the principle of deontic consequence (i.e., if α implies β, then $O\alpha$ implies $O\beta$).[30]

In his paper, Goble analyzes carefully how all these principles are related and what are the possible modifications one can make to try to answer the two aforementioned arguments. But more importantly, he argues that a logic that wishes to model normative conflicts and answer these arguments must satisfy four adequacy criteria [51, pp. 458–460, 470][31]:

1. consistency (a conflict must not entail a contradiction);
2. non-triviality (the logic must avoid deontic explosion);
3. the logic must satisfy the Smith argument;
4. the logic must satisfy the converse of (Agg).

[29] See also [50].
[30] See Castañeda [36, p. 13]. This also enables us to represent the distinction between fixed and derived obligations [cf. 4, p. 102].
[31] Another criterion might be added, namely that the formal system must not allow for the pragmatic oddity (cf. [35]). $\mathcal{CNR}$ answers this criterion (see [93]).

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To answer these concerns, Goble [51, pp. 451–452] argues that it is not the ought implies can principle which is problematic, nor the aggregation principle for that matter, but that it is rather the unrestricted distribution principle (Dist). Note, however, that in Goble's [51, p. 476] view, the principle (D) should be rejected since it is a 'no-conflict' principle which is not adequate to model obligations within real life. On this point, that (D) should be discarded, we disagree. $\mathcal{DDS}$ aims to model the Canadian legal discourse, and as such its presupposed consistency (after interpretation) is irrefutable.[32]

The first argument is attributed to Lemmon [75], who argued that aggregation and the ought implies can principle, together with the axiom schema (D) of standard deontic logic, should be rejected since they thwart the possibility of normative conflicts (see also [74]). Strictly speaking, this objection fails to affect $\mathcal{DDS}$ since it does not satisfy (Agg). Nonetheless, it remains that our version of (Can) together with the presumption of consistency (D) do imply the impossibility of normative conflicts.

Consider only the case of $\mathcal{CNR}$ for example. Assuming that two actions α and β cannot be performed together (i.e., assuming $\alpha \bullet \beta \underset{\mathsf{AL}}{\longrightarrow} *$), we obtain (1) by $(\Sigma_O^{\otimes})$.[33] Note that it follows from an instance of (Dist). This allows us to derive that the assumption $O\alpha \otimes O\beta$ fails within $\mathcal{CNR}$ (or that it is false in $\mathcal{OL}$), which is represented by (2). From (2) we can derive (3). As such, the following reformulation of Argument 1 does affect $\mathcal{DDS}$.

$$O\alpha \bullet \beta \underset{\mathsf{CNR}}{\longrightarrow} 0 \tag{1}$$

$$O\alpha \otimes O\beta \underset{\mathsf{CNR}}{\longrightarrow} 0 \tag{2}$$

$$c \otimes \big(c \multimap (O\alpha \otimes O\beta)\big) \underset{\mathsf{CNR}}{\longrightarrow} 0 \tag{3}$$

Argument 3. The ought implies can principle together with the presumption of normative consistency entail the impossibility of *unconditional* normative conflicts, or of conditional conflicting obligations that hold under the same context.

Thus formulated, however, this argument can be reduced to Argument 2 since the (Can) principle is derived from (Dist). That being said, it is important to see within the previous reasonings that the impossibility of normative conflicts happens when we assume that both $O\alpha$ and $O\beta$ hold *together* under the same circumstances. As a result, this objection fails to affect $\mathcal{DDS}$ since it violates the presumption of normative consistency. Legally speaking, conflicting obligations can only arise *a priori* within specific context. Otherwise, if there were unconditional conflicting obligations or (*a posteriori*) conflicting obligations that hold under the same conditions, the discourse would fail to be rational.[34] Recall that $\mathcal{DDS}$ takes place after interpreting the law. As such, the aforementioned normative conflicts are indeed impossible, and so Arguments 2 and 3 are actually arguments in favor of $\mathcal{DDS}$.

Now, consider the possibility of conditional conflicting obligations. Suppose that under circumstances c_1, $O\alpha$ holds, that in the context c_2, $O\beta$ holds, and that α and β cannot be accomplished together. There are four possibilities:

1. neither c_1 nor c_2 holds;
2. c_1 holds;
3. c_2 holds;
4. $c_1 \otimes c_2$ holds.

[32] See *2747-3174 Québec Inc. c. Québec (Régie des permis d'alcool)*, [1996] 3 RCS 919, paragraphe 207.
[33] See Appendix B for the proofs.
[34] As we said at the beginning of this paper, the Supreme Court of Canada assumes normative consistency as a criterion that enables us to determine whether or not the legal discourse is rational (cf. *2747-3174 Québec Inc. c. Québec (Régie des permis d'alcool)*, [1996] 3 RCS 919).

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The first case is unproblematic since $(c_1 \multimap O\alpha) \otimes (c_2 \multimap O\beta)$ entails neither $O\alpha$, $O\beta$ nor $O\alpha \otimes O\beta$.

The second and the third cases are analyzed similarly. Under the assumption that $c_1 \otimes ((c_1 \multimap O\alpha) \otimes (c_2 \multimap O\beta))$ holds, we can derive (4). Considering that $\alpha \bullet \beta$ is impossible to perform, we also have (5).

$$c_1 \otimes \big((c_1 \multimap O\alpha) \otimes (c_2 \multimap O\beta)\big) \underset{\mathsf{CNR}}{\longrightarrow} O\alpha \otimes (c_2 \multimap O\beta) \tag{4}$$

$$O\alpha \otimes (c_2 \multimap O\beta) \underset{\mathsf{CNR}}{\longrightarrow} O\beta^* \otimes (c_2 \multimap O\beta) \tag{5}$$

However, since c_2 is not assumed, $O\beta$ cannot be derived. As a result, under the context c_1, we obtain that α is obligatory although $O\beta$ holds under context c_2, or that β^* is obligatory although $O\beta$ holds under context c_2, which are perfectly consistent. Moreover, their interpretation are meaningful within the natural language: even though β is a conditional obligation that holds under context c_2, if we are under context c_1, then α is an actual obligation.

The fourth case might appear as the most problematic, but it actually is not. Of course, assuming that $c_1 \otimes c_2$ holds would allow us to derive $O\beta^* \otimes O\beta$. However, the presumption of consistency implies that after interpretation, the law must be consistent, and as such if $c_1 \otimes c_2$ holds, then one of the conflicting obligation will have priority over the other, and hence the conflict will be solved.

Prioritizing an obligation in case of a conflict was advocated by Alchourrón and Makinson [5]. More recently, van der Torre and Tan [115] argued that there are three different types of resolution for normative conflicts, depending upon the nature of the conflict. In a nutshell, they proposed that:

1. If a conflict arises from a contrary-to-duty, then the violated obligation is overshadowed by the contrary-to-duty obligation.
2. If there is a conflict between a *prima facie* and a conditional obligation, the *prima facie* ought is overshadowed by the conditional obligation.
3. If there is a conflict between two norms in a specific context, then one norm will override the other in that context.

Following van der Torre's and Tan's analysis, if $c_1 \otimes c_2$ holds, then one obligation will override the other, either by overshadowing it or by canceling it. As such, we need to specify which obligation is overridden by stipulating the conditions under which the obligations hold. For instance, if $O\alpha$ overrides $O\beta$ in the context $c_1 \otimes c_2$, then $O\alpha$ holds under the circumstances that c_1, c_2 and $c_2 \multimap O\beta$. Put differently, the assumption that $O\alpha$ overrides $O\beta$ can be translated by:

$$\big((c_1 \otimes c_2) \otimes (c_2 \multimap O\beta)\big) \multimap O\alpha$$

From this assumption and the context $(c_1 \otimes c_2) \otimes (c_2 \multimap O\beta)$, one can easily derive that $O\alpha$ is the obligation in force.

All things considered, $\mathcal{DDS}$ can properly model normative conflicts insofar as we properly specify the conditions under which the obligations hold (see [93] for details). As such, Arguments 1 and 2 fail to affect $\mathcal{DDS}$, although (Can), (Dist) and (D) are satisfied.

Let us now turn our attention to the four criteria advocated by Goble [51]. It was shown in [93] that $\mathcal{CNR}$, as a foundational framework, satisfies the criteria of consistency and non-triviality. This will be exemplified in Section 7, but for now simply note that consistency is satisfied when we specify the conditions under which the obligations hold, and non-triviality is satisfied given that $\mathcal{CNR}$ does not validate *ex falso sequitur quodlibet*.

The third criterion requires the satisfaction of the following inference pattern.

1.	It is obligatory that either α or β.
2.	It is obligatory that not-α.
$\therefore$	It is obligatory that β.

This inference pattern was initially presented by Horty [59,60] as the Smith argument, which is an argument in favor of the aggregation principle [cf. 51, p. 459]. The Smith argument goes along the following lines:

1.	Smith ought to fight in the army or else perform alternative service to his country.
2.	Smith ought to not fight in the army (say, because he needs to stay home to help his mother).
$\therefore$	Smith ought to perform alternative service to his country.

This can be seen as an argument in favor of the aggregation principle insofar as the reasoning is usually translated by:

$$\dfrac{\dfrac{O(p \vee q) \qquad O\neg p}{O((p \vee q) \wedge \neg p)}}{Oq}$$

Considering that $((p \vee q) \wedge \neg p) \supset q$, Oq follows from $O((p \vee q) \wedge \neg p)$. As we saw in Section 3, there is no such thing, in our view, as a *disjunctive* action. As such, assuming that only *actions* can be in the scope of a deontic operator, it follows that the aforementioned translation of the Smith argument is not accurate. Even though this translation is not available within $\mathcal{DDS}$, it happens that it is nonetheless possible to model the validity of the Smith argument. This, however, can be done *without* requiring any aggregation principle. The Smith argument can be translated within $\mathcal{DDS}$'s language by:

$$O\alpha \vee O\beta \tag{6}$$

$$O\alpha^* \tag{7}$$

$$O\beta \tag{8}$$

By (**D**) and (**P**) we have $O\alpha^* \underset{\mathsf{OL}}{\longrightarrow} \neg O\alpha$, and thus it is easy to show (9) using (Cart). By (cut), (9) and (10), we can easily derive (11).

$$O\alpha^* \wedge (O\alpha \vee O\beta) \underset{\mathsf{OL}}{\longrightarrow} \neg O\alpha \wedge (O\alpha \vee O\beta) \tag{9}$$

$$\neg O\alpha \wedge (O\alpha \vee O\beta) \underset{\mathsf{OL}}{\longrightarrow} O\beta \tag{10}$$

$$O\alpha^* \wedge (O\alpha \vee O\beta) \underset{\mathsf{OL}}{\longrightarrow} O\beta \tag{11}$$

As a result, $\mathcal{DDS}$ satisfies Goble's third criterion. It is noteworthy that Goble [50, p. 78] argued that although deontic explosion comes from *ex falso sequitur quodlibet*, the principle should not be discarded since otherwise the Smith argument cannot be validated. As it happens, $\mathcal{DDS}$ does satisfy the Smith argument while rejecting *ex falso sequitur quodlibet*.

The fourth criterion regards the following inference pattern.

1.	It is obligatory that α and β.
$\therefore$	It s obligatory that α.

This is the converse of the aggregation principle, translated by:

$$O(p \wedge q) \supset (Op \wedge Oq) \tag{12}$$

A first thing to point out is that Goble [51, p. 470] mentions that any logic satisfying (Dist) immediately satisfies the fourth criterion. In light of $\mathcal{DDS}$'s construction, this assumption is inaccurate: $\mathcal{DDS}$ satisfies (Dist) but does not satisfy (Agg) or its converse.

Secondly, Goble [51, p. 469] mentions that this inference pattern holds for any type of ought statement, including legal oughts. On this point we disagree. Contra Goble, we think that the fourth criterion does not represent a valid inference pattern for legal reasoning. This is mainly motivated by the type of action conjunction we use to model human actions. As we saw, an action conjunction $\alpha \bullet \beta$ represents the complex action 'α together with β', and this implies simultaneity. As such, it should be obvious to the reader that $O(\alpha \bullet \beta) \underset{\mathsf{OL}}{\longrightarrow} O\alpha$ does not hold. From a legal point of view, if the (simultaneous) conjunction of two actions is obligatory, it does not imply that the actions are obligatory individually (not simultaneously). Similarly, it is not because two actions α and β are obligatory independently that the conjunctive (simultaneous) action $\alpha \bullet \beta$ is obligatory. For instance, even though it is true that Paul has an obligation to help John and Jane who are drowning, this does not necessarily mean that he ought to help them *at the same time*.

Although neither (Agg) nor its converse are satisfied, it is noteworthy that $\mathcal{DDS}$ can still answer Goble's fourth criterion. Indeed, the converse of (Agg) can be obtained depending on the premises one adopts. For instance, if one was to assume (13), then by ($\Delta_{\otimes}$) or (Δ) one would be able to derive (14) or (15).

$$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} \underset{\mathsf{PAL}^{\otimes}}{\longrightarrow} \boldsymbol{\beta} \tag{13}$$

$$O(\alpha \bullet \beta) \underset{\mathsf{CNR}}{\longrightarrow} O\beta \tag{14}$$

$$O(\alpha \bullet \beta) \underset{\mathsf{OL}}{\longrightarrow} O\beta \tag{15}$$

In addition to the satisfaction of Goble's criteria, it is noteworthy that incidentally, $\mathcal{DDS}$'s construction enables us to answer Schotch's and Jennings's [105] objection against the modal system KD. Schotch and Jennings argued that the standard system fails to distinguish between the ought implies can principle and the presumption of normative consistency given the equivalence between the axiom schema (D) and $\neg O\bot$, which follows from the unrestricted use of the aggregation principle $O\varphi \wedge O\psi \vdash_{KD} O(\varphi \wedge \psi)$.

In our framework, these two principle are independent: while the presumption of normative consistency follows from (**D**) and (**P**), the ought implies can principle follows from the definition of (Σ_O) as a fibration. As such, whether or not we adopt ($\Diamond$), $\mathcal{DDS}$ answers the philosophical intuition that there is a distinction between the presumption of consistency of obligations and the ought implies can principle.

Various aggregation principles could be added to $\mathcal{DDS}$, but it is beyond the scope of this paper to analyze the consequences of adding one or another. Instead, simply notice that adding (**A**) as an axiom to $\mathcal{OL}$ (resp. $\mathcal{CNR}$) would blur the distinction between the presumption of normative consistency and the ought implies can principle. Indeed, $O\alpha \wedge O\alpha^* \underset{\mathsf{OL}}{\longrightarrow} \bot$ would be derivable from (**A**) and (Σ_O). Thus, adding (**A**) to $\mathcal{DDS}$ would reopen the door to Schotch's and Jennings's objection.

$$\frac{}{O\alpha \wedge O\beta \underset{\mathsf{OL}}{\longrightarrow} O(\alpha \bullet \beta)} \quad (\mathbf{A})$$

7. Discussion

The capacity of $\mathcal{CNR}$ to model contrary-to-duty reasoning and normative conflicts has already been discussed at length in [93]. For the purpose of this paper, we only exemplify how $\mathcal{DDS}$ satisfies Goble's first and second criteria of consistency and non-triviality. We also provide a discussion of the good Samaritan paradox and conclude with some example of analysis of legal reasoning.

7.1. Contrary-to-duty

Chisholm's [38] paradox shows that von Wright's [119] initial approach and, more generally, that monadic standard systems cannot model properly contrary-to-duty reasoning. Chisholm's puzzle amounts to the fact that the following set of propositions is inconsistent within a standard system, while it seems perfectly consistent in the natural language.

1. It is forbidden to exceed the speed limit on the highway (100 km/h).
2. If Paul exceeds the speed limit, then he ought to slowdown to 100 km/h.
3. Paul does not ought to slowdown to 100 km/h if he does not exceed the speed limit.
4. Paul exceeds the speed limit.

To solve Chisholm's paradox, a deontic logic must be able to provide an independent translation for each of these sentences (cf. [8,43,112,113,56]), and these translations must be consistent with each other. Moreover, one must be able to determine which obligation holds. In $\mathcal{DDS}$, Chisholm's puzzle is translated by:

$$O\alpha^{*} \qquad \mathsf{np/act} \tag{16}$$

$$\boldsymbol{\alpha} \multimap O\beta \qquad \mathsf{ap\backslash np/np} \tag{17}$$

$$\boldsymbol{\alpha^{*}} \multimap\sim O\beta \qquad \mathsf{ap\backslash np/np} \tag{18}$$

$$\boldsymbol{\alpha} \qquad \mathsf{ap} \tag{19}$$

These premises are non-redundant and give us the following result, which is perfectly consistent.

$$\big(\boldsymbol{\alpha^{*}} \multimap\sim O\beta\big) \otimes \big(O\alpha^{*} \otimes \big(\boldsymbol{\alpha} \otimes (\boldsymbol{\alpha} \multimap O\beta)\big)\big) \xrightarrow[\mathsf{CNR}]{} \big(\boldsymbol{\alpha^{*}} \multimap\sim O\beta\big) \otimes \big(O\alpha^{*} \otimes O\beta\big) \tag{20}$$

From these premises, we can also derive $O\alpha^{*} \otimes \sim \boldsymbol{\alpha^{*}}$ (the proof is left to the reader), which also is a desirable consequence: it is forbidden to exceed the speed limit on the highway but it is false that Paul is not exceeding the speed limit. As such, $\mathcal{DDS}$ is able to represent the fact that an obligation has been violated.

Nute and Yu [86, p. 6] argued that Chisholm's paradox follows when a logic allows for both *factual* and *deontic* detachment.[35] In standard deontic logic, (16) and (18) are translated respectively by $O\neg p$ and $O(\neg p \supset \neg q)$, which by deontic detachment leads to $O\neg q$. This contradicts Oq, which can be factually detached from the other premises. Let us see what happens if we assumed a form of deontic detachment and instead translated (18) by (21), expressing that ideally it is false that Paul ought to slow down. From these premises, we would obtain (22).

$$\sim O\beta \qquad \mathsf{np/np} \tag{21}$$

$$O\alpha^{*} \otimes \big(\sim O\beta \otimes \big(\boldsymbol{\alpha} \otimes (\boldsymbol{\alpha} \multimap O\beta)\big)\big) \xrightarrow[\mathsf{CNR}]{} O\alpha^{*} \otimes (\sim O\beta \otimes O\beta) \tag{22}$$

Note, however, that since $\mathcal{CNR}$ does not satisfy (Cart), it follows that the members of the conjunction cannot be detached. As such, neither $O\alpha^{*}$, $O\beta$, $\sim O\beta$ or $\sim O\beta \otimes O\beta$ is derivable. What we *do* have, from (22) and (cut), is (23). This result does *not* yield 0 since neither (Cart) nor (0) are available (see [93]). Hence, 0 can be seen as a failure, and even though some part of a package of normative premises can lead to a failure, it does not imply that the premises fail altogether. From a legal point of view, this is an interesting

[35] See [77] and [123] for a discussion on detachment.

property: it is not because some norms are inconsistent with each other that the whole legal system fails altogether.[36]

$$O\alpha^* \otimes \left(\sim O\beta \otimes \left(\boldsymbol{\alpha} \otimes (\boldsymbol{\alpha} \multimap O\beta)\right)\right) \underset{\mathsf{CNR}}{\longrightarrow} O\alpha^* \otimes 0 \tag{23}$$

In both cases, the premises do not lead to 0, and as such $\mathcal{DDS}$ satisfies Goble's criterion of consistency. With both translation, $\mathcal{DDS}$ is able to represent the intuitive consistency of Chisholm's set. However, these premises do not allow us to conclude which obligation is in force. To model that $O\beta$ should follow from this set, we need to appropriately translate the second premise. Van der Torre and Tan [115] argued that Chisholm's paradox is a case of factual defeasibility, where the contrary-to-duty obligation $O\beta$ overshadows the violated obligation $O\alpha^*$. As such, it is assumed that $O\beta$ holds under the circumstances that $\boldsymbol{\alpha}$, $O\alpha^*$ and $\boldsymbol{\alpha^*} \multimap\sim O\beta$ hold. Hence, (17) would rather be translated in $\mathcal{DDS}$ by (24), and we would thus get the desired result via (25).[37]

$$\left(O\alpha^* \otimes \left(\boldsymbol{\alpha} \otimes (\boldsymbol{\alpha^*} \multimap\sim O\beta)\right)\right) \multimap O\beta \tag{24}$$

$$\left(O\alpha^* \otimes \left(\boldsymbol{\alpha} \otimes (\boldsymbol{\alpha^*} \multimap\sim O\beta)\right)\right) \otimes \left(\left(O\alpha^* \otimes \left(\boldsymbol{\alpha} \otimes (\boldsymbol{\alpha^*} \multimap\sim O\beta)\right)\right) \multimap O\beta\right) \underset{\mathsf{CNR}}{\longrightarrow} O\beta \tag{25}$$

This last theorem exemplify the fact that $\mathcal{CNR}$ allows for *restricted* (factual) detachment, but only under specific circumstances. It was shown in [93] that $\mathcal{CNR}$ avoids the problem of augmentation [cf. 64], also known as the problem of strengthening the antecedent of a deontic conditional [cf. 3]. As a result, it does not allow for *unrestricted* detachment. With $\mathcal{CNR}$, (factual) detachment is only possible when we are under the specific context of the conditional obligation. If more information is added to the context, we need to evaluate whether or not the obligation still holds under that context. From a legal point of view, this is a desirable property: to assess whether or not an obligation holds under some conditions, one must interpret the law, and this is a task which is done by lawyers and judges. But more importantly, it is a task that cannot be done by logic alone [cf. 41].

7.2. Normative conflicts

Goble [50, p. 78] identified *ex falso sequitur quodlibet* as the "real culprit" for deontic explosion. Categorically speaking, *ex falso sequitur quodlibet* is represented by the axiom (0) and amounts to consider $\bot$ or 0 as initial objects. Despite this recognition, Goble [50,51] proposed to keep *ex falso sequitur quodlibet*, arguing that it is required to validate the Smith argument.

In $\mathcal{DDS}$, there is no need to keep *ex falso sequitur quodlibet* given that the Smith argument can be modeled through $\mathcal{OL}$. By constructing $\mathcal{CNR}$ as a symmetric closed deductive system, we reject (0) and therefore avoid deontic explosion, satisfying Goble's criterion of non-triviality.

Another interesting property of $\mathcal{DDS}$ is that it enables us to model that even though a norm is in place, it might happen in some situation that the norm does not hold anymore. Formally, this property is represented by the fact that the following does *not* hold for $\mathcal{CNR}$.

$$(\varphi \otimes \psi) \otimes \left[(\varphi \otimes \psi) \multimap\sim \psi\right] \longrightarrow 0$$

In a Cartesian deductive system, we would be able to obtain both ψ and $\sim \psi$ from $(\varphi\otimes\psi)\otimes[(\varphi\otimes\psi) \multimap\sim \psi]$. However, in $\mathcal{CNR}$, $\sim \psi$ is obtained from $(\varphi \otimes \psi)$, which can only be used once. To obtain $\psi\otimes \sim \psi$ from the

[36] This is also an interesting property if $\mathcal{DDS}$ is to have any repercussion on the field of artificial intelligence: some instructions may still be in force even though others fail.

[37] It can be objected that to obtain the desired result, one needs to make *ad hoc* modifications to the premises and do some previous inferences to encode the desired result within the premises. This objection has been dealt with in [93].

Please cite this article in press as: C. Peterson, The categorical imperative: Category theory as a foundation for deontic logic, Journal of Applied Logic (2014), http://dx.doi.org/10.1016/j.jal.2014.07.001

given premise, one would need to assume ψ twice. From a legal perspective, this allows us to model conflicts of norms that arise from specificity [cf. 115]. Even though some general norm holds, it might happen that in some context it is canceled and does not hold anymore.

7.3. The good Samaritan

The good Samaritan paradox was first presented by [99], but different formulations were offered by [85,8] and later [47]. The paradox goes along the following line of reasoning:

1. the good Samaritan ought to help the traveler who has been robbed;
2. if the good Samaritan helps the traveler who has been robbed, then the traveler has been robbed.

By deontic consequence, this imply that it ought to be that the traveler is robbed, which is a peculiar conclusion to be drawn. In $\mathcal{DDS}$, there are three plausible translations of the good Samaritan paradox. The first one is:

$$O(\alpha \bullet \beta) \qquad (\mathsf{np/act}) \tag{26}$$

$$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} \xrightarrow[\mathsf{PAL}^{\otimes}]{} \boldsymbol{\beta} \tag{27}$$

From this translation and $(\Delta_{\otimes})$, we obtain the undesired result that $O\beta$ follows from $O(\alpha \bullet \beta)$. This translation is, however, unlikely. The obligation for the good Samaritan is not to help the traveler *while* he is being robbed, but it is rather to help him afterwards. Hence, $O(\alpha \bullet \beta)$ is not an appropriate translation of the first premise given the properties of action conjunction.

The second translation available to us is:

$$O(\alpha \bullet \beta) \qquad (\mathsf{np/act}) \tag{28}$$

$$\boldsymbol{\alpha} \otimes \boldsymbol{\beta} \xrightarrow[\mathsf{PAL}^{\otimes}]{} \boldsymbol{\beta} \qquad \mathsf{ap} \tag{29}$$

Even though $O(\alpha \bullet \beta)$ is still not an appropriate translation of the first premise, it is noteworthy that with this translation, $(\Delta_{\otimes})$ cannot be applied, and as such we do not obtain a deduction arrow from $O(\alpha \bullet \beta)$ to $O\beta$.

This second translation could be a solution to a modified version of the good Samaritan paradox, which we call *the Hero paradox* (or, as we prefer, *Batman's paradox*):

1. Batman ought to help Robin, who is under attack by the Joker.
2. If Batman helps Robin, who is under attack by the Joker, then Robin is under attack by the Joker.

$\therefore$ The Joker ought to attack Robin.

The main difference between Batman's paradox and the good Samaritan paradox is that in the former the conjunctive action is simultaneous while in the latter it is not. As such, $O(\alpha \bullet \beta)$ could be a translation of the first premise in Batman's paradox. If it is, then the translation of the premises by (28) and (29) enables us to avoid the unlikely conclusion that the Joker has an obligation to attack Robin. That being said, one could easily argue that Batman's obligation is not to 'help Robin while he is under attack', but that it rather is to 'help Robin' under the circumstances that he is under attack. This is a plausible interpretation seeing that if a conjunctive action is obligatory, then it is expected that both parts of the conjunctive action can be accomplished by the same person. In this case, β should not be a part of Batman's obligations since it cannot be accomplished by Batman. The first premise could thus be translated by $\boldsymbol{\beta} \multimap O\alpha$, in which case, assuming that Robin is under attack, one could then conclude that Batman ought to help him.

The third translation of the good Samaritan paradox follows [8] and is more faithful to the meaning of the first premise in the natural language.

$$O\alpha \otimes \boldsymbol{\beta} \qquad \mathsf{(np/act)\backslash np/ap} \tag{30}$$

$$\boldsymbol{\alpha} \otimes \boldsymbol{\beta} \xrightarrow[\mathsf{PAL}^{\otimes}]{} \boldsymbol{\beta} \qquad \mathsf{ap} \tag{31}$$

Considering that $(\Delta_{\otimes})$ cannot be applied to (31), there is no arrow which can link the premises to $O\beta$.

7.4. Examples of analysis

We now show how $\mathcal{DDS}$ can help to model legal reasoning with the analysis of three examples.

Example 1. Considering that it is forbidden to remove an elm tree without treating the elm stump in a manner satisfactory to an inspector, it follows that it is forbidden to remove an elm tree while not treating the elm stump in a manner satisfactory to an inspector.[38]

This example is presented in an unconditional form. As such, it can be modeled through $\mathcal{OL}$ and the premise can be translated by (32). The inference is thus obtained by (33).

$$F(\alpha \ominus \beta) \qquad \mathsf{np/act} \tag{32}$$

$$F(\alpha \ominus \beta) \xrightarrow[\mathsf{OL}]{} F\big(\alpha \bullet \beta^{*}\big) \tag{33}$$

Example 2. It is forbidden to make tributyltetradecylphosphonium chloride ($F\alpha$), unless the following conditions are satisfied[39]:

1. the manufacture is for export only ($\boldsymbol{\beta}$);
2. the manufacturer has notified the Minister ($\boldsymbol{\gamma}$);
3. the manufacturer uses a fully contained process ($\boldsymbol{\eta}$).

This regulation could be translated either by (34) or (35).

$$F\big(\alpha \ominus \big(\beta \otimes (\gamma \otimes \eta)\big)\big) \tag{34}$$

$$F\alpha \otimes \big(\big(F\alpha \otimes \big(\boldsymbol{\beta} \otimes (\boldsymbol{\gamma} \otimes \boldsymbol{\eta})\big)\big) \multimap P_{s}\alpha\big) \tag{35}$$

However, Eq. (34) does not imply that it is permitted to make tributyltetradecylphosphonium chloride when the conditions are satisfied. Hence, (35) would be more accurate to the signification of the regulation within the natural language. It is a case of specificity, where $F\alpha$ is more general and is overruled in some context.

The regulation specifies that the permission holds under the circumstances that $F\alpha$ and $\boldsymbol{\beta} \otimes (\boldsymbol{\gamma} \otimes \boldsymbol{\eta})$, hence the conditional permission can be formulated by:

$$\big(F\alpha \otimes \big(\boldsymbol{\beta} \otimes (\boldsymbol{\gamma} \otimes \boldsymbol{\eta})\big)\big) \multimap P_{s}\alpha \tag{36}$$

Assuming a context where it is forbidden to make tributyltetradecylphosphonium chloride, but that it is permitted under the aforementioned circumstances, and that these circumstances are met, one can conclude

[38] Dutch Elm Disease Regulation, Manitoba Regulation 213/98.
[39] Tributyltetradecylphosphonium Chloride Regulations, SOR/2000-66.

that it is permitted to make tributyltetradecylphosphonium chloride by (1) and (cl).

$$\big(F\alpha \otimes (\boldsymbol{\beta} \otimes (\boldsymbol{\gamma} \otimes \boldsymbol{\eta}))\big) \otimes \big[\big(F\alpha \otimes (\boldsymbol{\beta} \otimes (\boldsymbol{\gamma} \otimes \boldsymbol{\eta}))\big) \multimap P_s\alpha\big] \underset{\mathsf{CNR}}{\longrightarrow} P_s\alpha \tag{37}$$

Example 3. The sign P-110-5 indicates that it is forbidden to make a U-turn, unless the vehicle is authorized.[40]

Assume that Paul, who did not drive an authorized vehicle, made a U-turn. In this case, one might want to be able to conclude from the previous regulation that it was forbidden for Paul to do so, and as such that he is liable. The first obvious translation for the regulation is (38), which says that it is usually forbidden to make a U-turn but it is permitted if the vehicle is authorized.

$$F\alpha \otimes (\boldsymbol{\beta} \multimap P_s\alpha) \tag{38}$$

This, however, would not allow us to conclude that it was forbidden for Paul to make a U-turn. In addition to this regulation, we would have in the context that $F\alpha$ but $\sim \boldsymbol{\beta}$, and since (Cart) is not satisfied, we would not be able to derive $F\alpha$. Another possible translation would be (39), which states that the interdiction is conditional to a context where there is a sign P-110-5 (c is of type d), and the U-turn is permitted in a context where there is a sign but the vehicle is authorized.

$$(c \multimap F\alpha) \otimes \big((c \otimes \boldsymbol{\beta}) \multimap P_s\alpha\big) \tag{39}$$

But again, this would not allow us to conclude that Paul should not have made a U-turn (i.e., that it was forbidden for him to do so), since the context would be $c\otimes \sim \boldsymbol{\beta}$. In order to be able to conclude that it was forbidden for Paul to make a U-turn, we need to assume that this interdiction holds under the circumstances that:

1. there is a sign;
2. the vehicle is unauthorized;
3. there is a conditional permission of making a U-turn when the vehicle is authorized.

$$\big(\big((c\otimes \sim \boldsymbol{\beta}) \otimes (\boldsymbol{\beta} \multimap P_s\alpha)\big) \multimap F\alpha\big) \otimes (\boldsymbol{\beta} \multimap P_s\alpha) \tag{40}$$

Assuming this together with $c\otimes \sim \boldsymbol{\beta}$, we would then be able to conclude $F\alpha$ from (1) and (cl).

Now assume that a police officer, who saw Paul make a U-turn, also made a U-turn to pursue Paul. In this case, we want to be able to conclude that the police officer was allowed to make the U-turn. The solution is, as in the case of the interdiction, to specify adequately the conditions under which the permission holds. The conditional permission is meant to hold when:

1. there is a sign;
2. the vehicle is authorized;
3. there is a conditional interdiction of making a U-turn when the vehicle is unauthorized.[41]

$$\big(\big((c \otimes \boldsymbol{\beta}) \otimes (\sim \boldsymbol{\beta} \multimap F\alpha)\big) \multimap P_s\alpha\big) \otimes (\sim \boldsymbol{\beta} \multimap F\alpha) \tag{41}$$

[40] Quebec Regulations on Road Signs, RRQ, c C-24.2, r 41.
[41] We could also use a general interdiction instead of a conditional one, or an interdiction conditional to $c\otimes \sim \boldsymbol{\beta}$.

Together with $c \otimes \boldsymbol{\beta}$, this yields $P_s\alpha$. This third example shows that interpreting the law is a crucial part of legal reasoning. To do a proper legal inference, one must first determine the context and the normative propositions that hold under that context. As such, there is a human component to legal reasoning, which consists of assessing the appropriate premises of the argument.

8. Conclusion

Summing up, we introduced a deontic deductive system $\mathcal{DDS}$ built upon an action logic $\mathcal{AL}$, a propositional action logic $\mathcal{PAL}$, an obligation logic $\mathcal{OL}$ and a logic for conditional normative reasoning $\mathcal{CNR}$. An interesting fact about $\mathcal{DDS}$ is that all its components have a different structure. From a philosophical point of view, this is not only a desirable property, but it is foremost a necessary one. To model normative reasoning, one must be able to reason with facts, with actions, with facts about actions, with norms and with conditional norms. The complexity of our natural language relies namely upon the fact that some of its parts do not share the same structure. Reasoning with conditional obligations does not require the same characteristics as reasoning with unconditional obligations. Similarly, reasoning about what actions are is not the same thing as reasoning about which actions are true. By incorporating different structures, $\mathcal{DDS}$ allows for the representation of the complexity of our natural normative language.

Another interesting fact about $\mathcal{DDS}$ is that it possesses a familiar structure. The rules that relate the different fragments were defined as fibrations. They allow a coherent presentation of $\mathcal{DDS}$ and they insure that all its fragments share a common structure. This is interesting given that fibrations are common structures in mathematics and in type theory, notwithstanding the recent developments made in homotopy type theory. From an epistemological point of view, defining $\mathcal{DDS}$ with the help of fibrations hints at the possibility that the structure of our natural language might be more familiar than we think with some mathematical structures.

The main motivation of this article was to introduce a deontic logic that can be used to model the Canadian legal discourse. As such, it was constructed upon some of its fundamental characteristics, including the presumption of normative consistency, which insures its rationality. We showed how $\mathcal{DDS}$ can be used to solve some paradoxes of deontic logic and provided some examples of analysis of legal reasoning. Building on previous work, we used category theory as a proof-theoretical foundational framework for logic, and we showed how some problems for deontic logic can be solved by using different types of deductive systems. For future research, we intend to study how $\mathcal{DDS}$ could be extended with epistemic modalities. Furthermore, we intend to determine how it could be applied to cognitive sciences and artificial intelligence, examining how it can model learning processes and uncertain knowledge.

All things considered, we hope to have convinced the reader that a categorical analysis of logic is relevant to philosophical issues, hence the categorical imperative.

Appendix A

Definition 19. (See Mac Lane [79], pp. 7–8.) A category $\mathcal{C}$ is composed of (1) $\mathcal{C}$-*objects*, (2) $\mathcal{C}$-*arrows*, (3) an operation assigning to each arrow a domain and a codomain within the $\mathcal{C}$-objects, (4) composition of arrow gf for each pair $f : x \longrightarrow y$ and $g : y \longrightarrow z$ that respects associativity, i.e. $h(gf) = (hg)f$ and (5) an identity arrow 1_y for each $\mathcal{C}$-object y such that $1_y f = f$ and $g1_y = g$ for each pair $f : x \longrightarrow y$ and $g : y \longrightarrow z$ (thus respecting the identity laws).

Definition 20. (See Mac Lane [79], pp. 161–162.) A *monoidal category* is a category equipped with a tensor product $\otimes$ and a unit object I. The tensor product is associative, and hence there is an arrow $a_{x,y,z}$ for each $\mathcal{C}$-objects x, y and z, and the unit object respects l_x and r_x for each $\mathcal{C}$-objects x. These arrows are natural

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isomorphisms. The tensor product and the unit have to respect the triangle and pentagon identities.

$$a_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$
$$l_x : I \otimes x \longrightarrow x$$
$$r_x : x \otimes I \longrightarrow x$$

Definition 21. (See Mac Lane [79], p. 251.) A *symmetric monoidal category* is a monoidal category with a natural isomorphism $\beta_{x,y} : x \otimes y \longrightarrow y \otimes x$ which is its own inverse. It has to satisfy the hexagon identities.

Definition 22. (See Mac Lane [79], p. 184.) A *closed monoidal category* is a monoidal category $\mathcal{C}$ where the tensor product $- \otimes x : \mathcal{C} \longrightarrow \mathcal{C}$ has right adjoint $(-)^x : \mathcal{C} \longrightarrow \mathcal{C}$.

Definition 23. (See Barr [10,11].) A **-autonomous category* is a symmetric closed category with a dualizing object $*$, that is, an object such that $x \longrightarrow *^{*^x}$ is an isomorphism for every object x.

Definition 24. (See Kelly and Laplaza [67].) A *compact closed category* is a $*$-autonomous category such that $y^x \longrightarrow *^x \otimes y$ is an isomorphism.

Definition 25. (See Mac Lane [79], p. 13.) A *functor* $\mathcal{C} \xrightarrow{\Phi} \mathcal{B}$ is a morphism between two categories such that:

1. there is $\Phi(c)$ in $\mathcal{B}$ for each c in $\mathcal{C}$;
2. there is $\Phi(c_1) \xrightarrow{\Phi(f)} \Phi(c_2)$ in $\mathcal{B}$ for each $c_1 \xrightarrow{f} c_2$ in $\mathcal{C}$;
3. $\Phi(1_c) = 1_{\Phi(c)}$;
4. $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$.

Definition 26. (See Jacobs [63], p. 27.) Let $\mathcal{C} \xrightarrow{\Phi} \mathcal{B}$ be a functor. Then, $c_1 \xrightarrow{f} c_2$ in $\mathcal{C}$ is *Cartesian over* $b_1 \xrightarrow{u} b_2$ in $\mathcal{B}$ when:

1. $\Phi(f) = u$;
2. for every $c_3 \xrightarrow{g} c_2$ in $\mathcal{C}$ such that $\Phi(g) = u \circ w$ for some $\Phi(c_3) \xrightarrow{w} b_1$, there is one and only one $c_3 \xrightarrow{h} c_1$ such that $g = f \circ h$.

Definition 27. (See Jacobs [63], p. 27.) $\mathcal{C} \xrightarrow{\Phi} \mathcal{B}$ is a *fibration* when for every c_i in $\mathcal{C}$ and $b_i \xrightarrow{u} \Phi(c_i)$ in $\mathcal{B}$, there is f in $\mathcal{C}$ Cartesian over u.

Definition 28. (See Mac Lane [79], p. 14.) A functor $\mathcal{C} \xrightarrow{\Phi} \mathcal{B}$ is *full* when for every $\mathcal{C}$-objects c, c' and $\mathcal{B}$-arrow $g : \Phi(c) \longrightarrow \Phi(c')$, there is $f : c \longrightarrow c'$ such that $g = \Phi(f)$.

Appendix B

We now provide the proofs of most of the theorems one can find within the paper. We omit the steps using associativity and symmetry. We follow the theorem numbering.

1.

$$\dfrac{\dfrac{}{\alpha \bullet \beta \xrightarrow[\mathsf{AL}]{} *}\ (\mathrm{H})}{O\alpha \bullet \beta \xrightarrow[\mathsf{CNR}]{} 0}\ (\Sigma_O^{\otimes})$$

2.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(H)}
\UnaryInfC{$\alpha \bullet \beta \xrightarrow[\mathsf{AL}]{} *$}
\RightLabel{(cl)}
\UnaryInfC{$\alpha \xrightarrow[\mathsf{AL}]{} * \ominus \beta$}
\UnaryInfC{$\alpha \xrightarrow[\mathsf{AL}]{} \beta^*$}
\RightLabel{$(\Sigma_O^{\otimes})$}
\UnaryInfC{$O\alpha \xrightarrow[\mathsf{CNR}]{} O\beta^*$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$O\beta \xrightarrow[\mathsf{CNR}]{} O\beta$}
\RightLabel{(t)}
\BinaryInfC{$O\alpha \otimes O\beta \xrightarrow[\mathsf{CNR}]{} O\beta^* \otimes O\beta$}
\AxiomC{$\vdots$}
\RightLabel{(NC)}
\UnaryInfC{$O\beta^* \otimes O\beta \xrightarrow[\mathsf{CNR}]{} 0$}
\RightLabel{(cut)}
\BinaryInfC{$O\alpha \otimes O\beta \xrightarrow[\mathsf{CNR}]{} 0$}
\end{prooftree}
$$

3.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$c \multimap (O\alpha \otimes O\beta) \xrightarrow[\mathsf{CNR}]{} c \multimap (O\alpha \otimes O\beta)$}
\RightLabel{(cl)}
\UnaryInfC{$c \otimes (c \multimap (O\alpha \otimes O\beta)) \xrightarrow[\mathsf{CNR}]{} O\alpha \otimes O\beta$}
\AxiomC{$\vdots$}
\RightLabel{(from (2))}
\UnaryInfC{$O\alpha \otimes O\beta \xrightarrow[\mathsf{CNR}]{} 0$}
\RightLabel{(cut)}
\BinaryInfC{$c \otimes (c \multimap (O\alpha \otimes O\beta)) \xrightarrow[\mathsf{CNR}]{} 0$}
\end{prooftree}
$$

4.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$c_1 \multimap O\alpha \xrightarrow[\mathsf{CNR}]{} c_1 \multimap O\alpha$}
\RightLabel{(cl)}
\UnaryInfC{$c_1 \otimes (c_1 \multimap O\alpha) \xrightarrow[\mathsf{CNR}]{} O\alpha$}
\AxiomC{}
\UnaryInfC{$c_2 \multimap O\beta \xrightarrow[\mathsf{CNR}]{} c_2 \multimap O\beta$}
\RightLabel{(t)}
\BinaryInfC{$c_1 \otimes ((c_1 \multimap O\alpha) \otimes (c_2 \multimap O\beta)) \xrightarrow[\mathsf{CNR}]{} O\alpha \otimes (c_2 \multimap O\beta)$}
\end{prooftree}
$$

5.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\RightLabel{(from the assumption in the proof of (1))}
\UnaryInfC{$\alpha \xrightarrow[\mathsf{AL}]{} \beta^*$}
\RightLabel{$(\Sigma_O^{\otimes})$}
\UnaryInfC{$O\alpha \xrightarrow[\mathsf{CNR}]{} O\beta^*$}
\AxiomC{}
\UnaryInfC{$c_2 \multimap O\beta \xrightarrow[\mathsf{CNR}]{} c_2 \multimap O\beta$}
\RightLabel{(t)}
\BinaryInfC{$O\alpha \otimes (c_2 \multimap O\beta) \xrightarrow[\mathsf{CNR}]{} O\beta^* \otimes (c_2 \multimap O\beta)$}
\end{prooftree}
$$

10.

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(1)}
\UnaryInfC{$\neg O\alpha \xrightarrow[\mathsf{OL}]{} O\alpha \supset \bot$}
\LeftLabel{(cl)}
\UnaryInfC{$\neg O\alpha \wedge O\alpha \xrightarrow[\mathsf{OL}]{} \bot$}
\AxiomC{}
\LeftLabel{(0)}
\UnaryInfC{$\bot \xrightarrow[\mathsf{OL}]{} O\beta$}
\LeftLabel{(cut)}
\BinaryInfC{$\neg O\alpha \wedge O\alpha \xrightarrow[\mathsf{OL}]{} O\beta$}
\LeftLabel{(cl)}
\UnaryInfC{$O\alpha \xrightarrow[\mathsf{OL}]{} \neg O\alpha \supset O\beta$}
\AxiomC{}
\LeftLabel{(!)}
\UnaryInfC{$\neg O\alpha \xrightarrow[\mathsf{OL}]{} \top$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$O\beta \wedge \top \xrightarrow[\mathsf{OL}]{} O\beta \wedge \top$}
\RightLabel{(Cart)}
\UnaryInfC{$O\beta \wedge \top \xrightarrow[\mathsf{OL}]{} O\beta$}
\RightLabel{(cl)}
\UnaryInfC{$\top \xrightarrow[\mathsf{OL}]{} O\beta \supset O\beta$}
\RightLabel{(cut)}
\BinaryInfC{$\neg O\alpha \xrightarrow[\mathsf{OL}]{} O\beta \supset O\beta$}
\RightLabel{(cl)}
\UnaryInfC{$\neg O\alpha \wedge O\beta \xrightarrow[\mathsf{OL}]{} O\beta$}
\RightLabel{(cl)}
\UnaryInfC{$O\beta \xrightarrow[\mathsf{OL}]{} \neg O\alpha \supset O\beta$}
\LeftLabel{(coCart)}
\BinaryInfC{$(O\alpha \vee O\beta) \xrightarrow[\mathsf{OL}]{} \neg O\alpha \supset O\beta$}
\LeftLabel{(cl)}
\UnaryInfC{$\neg O\alpha \wedge (O\alpha \vee O\beta) \xrightarrow[\mathsf{OL}]{} O\beta$}
\end{prooftree}
$$

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20.

$$\dfrac{\dfrac{}{\boldsymbol{\alpha}^* \multimap\sim O\beta \xrightarrow[\mathsf{CNR}]{} \boldsymbol{\alpha}^* \multimap\sim O\beta}\,(1) \qquad \dfrac{\dfrac{}{O\alpha^* \xrightarrow[\mathsf{CNR}]{} O\alpha^*}\,(1) \quad \dfrac{\dfrac{}{\boldsymbol{\alpha}\multimap O\beta \xrightarrow[\mathsf{CNR}]{} \boldsymbol{\alpha}\multimap O\beta}\,(1)}{\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta)\xrightarrow[\mathsf{CNR}]{} O\beta}\,(\mathrm{cl})}{O\alpha^*\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta))\xrightarrow[\mathsf{CNR}]{} O\alpha^*\otimes O\beta}\,(\mathrm{t})}{(\boldsymbol{\alpha}^* \multimap\sim O\beta)\otimes(O\alpha^*\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta)))\xrightarrow[\mathsf{CNR}]{}(\boldsymbol{\alpha}^* \multimap\sim O\beta)\otimes(O\alpha^*\otimes O\beta)}\,(\mathrm{t})$$

22.

$$\dfrac{\dfrac{}{O\alpha^* \xrightarrow[\mathsf{CNR}]{} O\alpha^*}\,(1) \qquad \dfrac{\dfrac{}{\sim O\beta \xrightarrow[\mathsf{CNR}]{} \sim O\beta}\,(1) \quad \dfrac{\dfrac{}{\boldsymbol{\alpha}\multimap O\beta \xrightarrow[\mathsf{CNR}]{} \boldsymbol{\alpha}\multimap O\beta}\,(1)}{\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta)\xrightarrow[\mathsf{CNR}]{} O\beta}\,(\mathrm{cl})}{\sim O\beta\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta))\xrightarrow[\mathsf{CNR}]{} \sim O\beta\otimes O\beta}\,(\mathrm{t})}{O\alpha^*\otimes(\sim O\beta\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta)))\xrightarrow[\mathsf{CNR}]{} O\alpha^*\otimes(\sim O\beta\otimes O\beta)}\,(\mathrm{t})$$

23.

$$\dfrac{\dfrac{\vdots}{O\alpha^*\otimes(\sim O\beta\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta)))\xrightarrow[\mathsf{CNR}]{} O\alpha^*\otimes(\sim O\beta\otimes O\beta)}\,(\text{from (22)}) \qquad \dfrac{\dfrac{}{O\alpha^* \xrightarrow[\mathsf{CNR}]{} O\alpha^*}\,(1) \quad \dfrac{\dfrac{}{\sim O\beta \xrightarrow[\mathsf{CNR}]{} O\beta\multimap 0}\,(1)}{\sim O\beta\otimes O\beta\xrightarrow[\mathsf{CNR}]{} 0}\,(\mathrm{cl})}{O\alpha^*\otimes(\sim O\beta\otimes O\beta)\xrightarrow[\mathsf{CNR}]{} O\alpha^*\otimes 0}\,(\mathrm{t})}{O\alpha^*\otimes(\sim O\beta\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}\multimap O\beta)))\xrightarrow[\mathsf{CNR}]{} O\alpha^*\otimes 0}\,(\mathrm{cut})$$

25.

$$\dfrac{\dfrac{}{(O\alpha^*\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}^*\multimap\sim O\beta)))\multimap O\beta \xrightarrow[\mathsf{CNR}]{} (O\alpha^*\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}^*\multimap\sim O\beta)))\multimap O\beta}\,(1)}{(O\alpha^*\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}^*\multimap\sim O\beta)))\otimes(O\alpha^*\otimes(\boldsymbol{\alpha}\otimes(\boldsymbol{\alpha}^*\multimap\sim O\beta)))\multimap O\beta \xrightarrow[\mathsf{CNR}]{} O\beta}\,(\mathrm{cl})$$

33.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{(from the proofs below)}}{\alpha\ominus\beta\cong\alpha\bullet\beta^*}}{(\alpha\ominus\beta)^*\cong(\alpha\bullet\beta^*)^*}}{O(\alpha\ominus\beta)^*\xrightarrow[\mathsf{OL}]{} O(\alpha\bullet\beta^*)^*}\,(\Delta)}{F(\alpha\ominus\beta)\xrightarrow[\mathsf{OL}]{} F(\alpha\bullet\beta^*)}\,(\text{def. } F)$$

We now list some other useful deductions. First, it is quite easy to show from (cpt1) and (cpt2) that $\alpha\bullet\beta^*\cong\alpha\ominus\beta$.

$$\dfrac{\dfrac{\dfrac{}{\alpha\bullet\beta^*\xrightarrow[\mathsf{AL}]{}\alpha\bullet\beta^*}\,(1)}{\alpha\bullet\beta^*\xrightarrow[\mathsf{AL}]{}\beta^*\bullet\alpha}\,(\mathrm{b}) \qquad \dfrac{}{\beta^*\bullet\alpha\xrightarrow[\mathsf{AL}]{}\alpha\ominus\beta}\,(\mathrm{cpt1})}{\alpha\bullet\beta^*\xrightarrow[\mathsf{AL}]{}\alpha\ominus\beta}\,(\mathrm{cut})$$

$$\dfrac{(\text{cpt2})\ \dfrac{}{\alpha\ominus\beta\underset{\mathsf{AL}}{\longrightarrow}\beta^*\bullet\alpha} \qquad \dfrac{\dfrac{}{\beta^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\beta^*\bullet\alpha}\ (1)}{\beta^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\beta^*}\ (\text{b})}{\alpha\ominus\beta\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\beta^*}\ (\text{cut})$$

Moreover, it is possible to show that $\alpha{\cong}\alpha^{**}$. The first part of the proof follows from the definition of the dual object by $\alpha^* =_{def} * \ominus \alpha$.

$$\dfrac{\dfrac{\dfrac{(\text{b})\ \dfrac{(1)\ \dfrac{}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\alpha^*}}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha^*\bullet\alpha} \qquad \dfrac{\dfrac{\dfrac{}{\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha^*}\ (1)}{\alpha^*\underset{\mathsf{AL}}{\longrightarrow} * \ominus\alpha}}{\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}*}\ (\text{cl})}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}*}\ (\text{cut})}{\alpha\underset{\mathsf{AL}}{\longrightarrow} * \ominus\alpha^*}\ (\text{cl})}{\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha^{**}}$$

From this proof, one can also see that $\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}*$. The other part is a bit more tricky.

$$(\text{cut})\ \dfrac{(\text{t})\ \dfrac{\dfrac{(\text{from previous proof})}{\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha^{***}} \qquad (1)\ \dfrac{}{\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha}}{\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha^{***}\bullet\alpha} \qquad \dfrac{}{\alpha^{***}\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha^{**}}\ (\text{cpt2})}{\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha^{**}}$$

$$(\text{cut})\ \dfrac{(\text{b})\ \dfrac{(1)\ \dfrac{}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\alpha^*}}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha^*\bullet\alpha} \qquad \dfrac{\vdots}{\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha^{**}}}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha^{**}}$$

$$\dfrac{(\text{cut})\ \dfrac{(\text{cl})\ \dfrac{(\text{r})\ \dfrac{(1)\ \dfrac{}{\alpha\bullet *\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet *}}{\alpha\bullet *\underset{\mathsf{AL}}{\longrightarrow}\alpha}}{*\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha} \qquad (\text{cpt2})\ \dfrac{}{\alpha\ominus\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha^*\bullet\alpha}}{*\underset{\mathsf{AL}}{\longrightarrow}\alpha^*\bullet\alpha} \qquad \dfrac{\dfrac{}{\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha^*\bullet\alpha}\ (1)}{\alpha^*\bullet\alpha\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\alpha^*}\ (\text{b})}{*\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\alpha^*}\ (\text{cut})$$

$$\dfrac{(\text{r})\ \dfrac{(1)\ \dfrac{}{\alpha^{**}\underset{\mathsf{AL}}{\longrightarrow}\alpha^{**}}}{\alpha^{**}\underset{\mathsf{AL}}{\longrightarrow}\alpha^{**}\bullet *} \qquad \dfrac{\dfrac{\dfrac{\vdots}{*\underset{\mathsf{AL}}{\longrightarrow}\alpha\bullet\alpha^*} \qquad \dfrac{\vdots}{\alpha\bullet\alpha^*\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha^{**}}}{*\underset{\mathsf{AL}}{\longrightarrow}\alpha\ominus\alpha^{**}}\ (\text{cut})}{\alpha^{**}\bullet *\underset{\mathsf{AL}}{\longrightarrow}\alpha}\ (\text{cl})}{\alpha^{**}\underset{\mathsf{AL}}{\longrightarrow}\alpha}\ (\text{cut})$$

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From this proof, we also get $* \cong \alpha \bullet \alpha^*$. Although it might seem curious at first glance, it happens in $\mathcal{AL}$ that we have $*{\cong}*^*$. This might seem peculiar since in terms of truth value, this would mean that $\top \cong \neg\top$. This is actually an interesting property of compact deductive systems.

$$\dfrac{\dfrac{\dfrac{}{*^* \xrightarrow[\mathsf{AL}]{} *^*}\ (1)}{*^* \xrightarrow[\mathsf{AL}]{} *^* \bullet *}\ (\mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{}{*^* \bullet * \xrightarrow[\mathsf{AL}]{} *^* \bullet *}\ (1)}{*^* \bullet * \xrightarrow[\mathsf{AL}]{} * \bullet *^*}\ (\mathrm{b}) \qquad \dfrac{\dfrac{\dfrac{}{*^* \xrightarrow[\mathsf{AL}]{} *^*}\ (1)}{*^* \xrightarrow[\mathsf{AL}]{} * \ominus *}}{* \bullet *^* \xrightarrow[\mathsf{AL}]{} *}\ (\mathrm{cl})}{*^* \bullet * \xrightarrow[\mathsf{AL}]{} *}\ (\mathrm{cut})}{*^* \xrightarrow[\mathsf{AL}]{} *}\ (\mathrm{cut})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{* \bullet * \xrightarrow[\mathsf{AL}]{} * \bullet *}\ (1)}{* \bullet * \xrightarrow[\mathsf{AL}]{} *}\ (\mathrm{r})}{* \xrightarrow[\mathsf{AL}]{} * \ominus *}}{* \xrightarrow[\mathsf{AL}]{} *^*}\ (\mathrm{cl})$$

Here, one can easily see why formulas of $\mathcal{AL}$ should not be interpreted in terms of declarative statements. The formula $\alpha \bullet \alpha^*$ is neither a contradiction nor a tautology, and $*$ is neither *true* nor *false*.

Finally, one can prove that $\alpha{\cong}\beta$ if and only if $\alpha^*{\cong}\beta^*$. This will help when we reason with interdictions. The proof is quite straight forward. From left to right, assume $\alpha{\cong}\beta$, hence $\alpha{\cong}\beta^{**}$, and thus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \xrightarrow[\mathsf{AL}]{} \beta^{**}}{\alpha \xrightarrow[\mathsf{AL}]{} * \ominus \beta^*}}{\beta^* \bullet \alpha \xrightarrow[\mathsf{AL}]{} *}}{\beta^* \xrightarrow[\mathsf{AL}]{} * \ominus \alpha}}{\beta^* \xrightarrow[\mathsf{AL}]{} \alpha^*}$$

By the same reasoning, we obtain $\alpha^* \xrightarrow[\mathsf{AL}]{} \beta^*$ from $\alpha^{**}{\cong}\beta$. From right to left, assume $\alpha^*{\cong}\beta^*$, hence we have (omitting symmetry):

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha^* \xrightarrow[\mathsf{AL}]{} \beta^*}{\alpha^* \xrightarrow[\mathsf{AL}]{} * \ominus \beta}}{\beta \bullet \alpha^* \xrightarrow[\mathsf{AL}]{} *}}{\beta \xrightarrow[\mathsf{AL}]{} * \ominus \alpha^*}}{\beta \xrightarrow[\mathsf{AL}]{} \alpha^{**}}$$

And since we know that $\alpha{\cong}\alpha^{**}$, it follows from (cut) that $\beta \xrightarrow[\mathsf{AL}]{} \alpha$. We obtain $\alpha \xrightarrow[\mathsf{AL}]{} \beta$ by the same reasoning.

Finally, $\sim 1 \longrightarrow 0$, $0 \longrightarrow {\sim}1$ and $1 \longrightarrow {\sim}0$ are $\mathcal{CNR}$-arrows. We omit the steps for associativity and symmetry.

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$$\dfrac{\mathrm{(cut)}\ \dfrac{\mathrm{(r)}\ \dfrac{\mathrm{(1)}\ \dfrac{}{1 \multimap 0 \longrightarrow 1 \multimap 0}}{1 \multimap 0 \longrightarrow 1 \otimes (1 \multimap 0)} \qquad \mathrm{(cl)}\ \dfrac{\mathrm{(1)}\ \dfrac{}{1 \multimap 0 \longrightarrow 1 \multimap 0}}{1 \otimes (1 \multimap 0) \longrightarrow 0}}{1 \multimap 0 \longrightarrow 0}}{\sim 1 \longrightarrow 0}$$

$$\dfrac{\mathrm{(cl)}\ \dfrac{\mathrm{(r)}\ \dfrac{\mathrm{(1)}\ \dfrac{}{1 \otimes 0 \longrightarrow 1 \otimes 0}}{1 \otimes 0 \longrightarrow 0}}{1 \longrightarrow 0 \multimap 0}}{1 \longrightarrow \sim 0}$$

By contraposition we thus obtain $0 \longrightarrow \sim 1$.

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